Groups of Polynomial Growth Learning Seminar
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Mondays 1:30 pm - 2:30 pm

Outline
I) Setting
II) Key Features
III) Some Results
} General Overview
IV) Basic Lie Theory } Some Prereq.
V) Geometry
VI) Structure Theory
} The meat.

Appendix
i) Questions
ii) Explicit Examples
iii) Bib.

Main Ref: Analysis on Lie Groups with Polynomial Growth-
Dungey, ter Elst, Robinson

*Groups are always connected and often simply connected.

I) Setting

$(G, \mathbb{X}, d\mu, d)$

G = a Lie group
$\mathbb{X} = \{X_1, \ldots, X_m\}$ = a Hörmander basis
$d\mu$ = Haar measure
d_{cc} = Carnot – Carathéodory (CC) - metric induced by $\mathbb{X}$.

Group of Polynomial Growth :

Let B_r = cc-metric ball centered at $e \in G$ of radius r
$|B_r|$ = $d\mu$-volume of B_r

↳ G is a GoPG $\Leftrightarrow$ $\exists\, D \in \mathbb{N}$, $c, C > 0$ s.t.

$$c r^D \leq |B_r| \leq C r^D, \quad r \geq 1.$$

D does not depend on the choice of $\mathbb{X}$ (hence on $m = \# \mathbb{X}$ nor d_{cc})
↳ D is intrinsic to G.

Examples:

i) compact groups and nilpotent groups are G.P.G.

ii) Non-unimodular always have exponential growth

iii) PG $\Rightarrow$ unimodular but unimodular $\not\Rightarrow$ PG

iv) Non-compact semi-simple Lie groups are unimodular but of exponential Growth.

v) Harmonic AN groups have exponential growth.

II) Key Features

1) Group action $G \circlearrowright G$ constrains volume growth
2) May study global properties of G using heat equation methods

$L = \partial_t \longrightarrow |h_t| \lesssim t^{-\frac{1}{2}} |B_{\sqrt{t}}|^{-1} e^{-|x|^2/ct}, \ t > 0$

sub-Laplacian $\quad \|h_t\|_{L^p} \lesssim t^{-\frac{1}{2}} |B_{\sqrt{t}}|^{-\frac{1}{p'}}, \ t > 0$

3) Natural setting to study complex coefficient invariant 2nd order subelliptic operators on G.
4) Rich structure theory:

$G = M \rtimes Q$, $Q \rightsquigarrow Q_N$ = nilshadow

cpt, solvable, nilpotent, diffeo. to Q.

- $G \rightsquigarrow$ "cylindrical manifold" $M \times Q_N$
  - Heat equation analysis $\rightsquigarrow \mathbb{T}^d \times \mathbb{R}^m$
  - H on $G \rightsquigarrow \hat{H}$ on Q_N

2nd order invariant subelliptic

$\hat{H}$ governs asymptotics of H. Uses homogenization theory.

  - $K, \hat{K}$ semigroup kernels for $H, \hat{H}$

$\hookrightarrow \quad |K_t(mq) - \hat{K}_t(q)| \lesssim t^{-1/2} |B_{\sqrt{t}}|^{-1/2}$

III) Some Results

Heat Kernel Estimates:

Proposition (Saloff-Coste, Varopolous)

G a GoPG; $\mathfrak{X} = \{X_1, \dots, X_m\}$ = Hörmander basis

h_t = heat kernel of $\mathcal{L} - \partial_t \triangleq \sum X_i^2 - \partial_t = 0$

$\hookrightarrow$

$$|X_i \partial_t^k h_t(x)| \lesssim |B_{\sqrt{t}}|^{-1} t^{-k-1/2} e^{-\frac{|x|^2}{ct}}, \quad (t,x) \in \mathbb{R}_{>0} \times G$$

$$h_t(x) \gtrsim |B_{\sqrt{t}}|^{-1} e^{-\frac{c|x|^2}{t}}, \quad (t,x) \in \mathbb{R}_{>0} \times G$$

Sobolev Inequality

Theorem (Varapolous)

G a GoPG

d = local dimension $\quad |B_r| \sim r^d$ for $0 < r < 1$

D = dimension at ∞ $\quad |B_r| \sim r^D$ for $1 < r < \infty$

$\alpha \in \mathbb{N}_{>0}$, $1 \le p < \infty$, $n \in [d, D]$, $\alpha p < n$, $q = \frac{np}{n - \alpha p}$

$\hookrightarrow$ $\|f\|_q \le c \|f\|_{p,\alpha}$, $\quad f \in C_0^\infty(G)$

$\|f\|_{p,0} = \|f\|_p$, $\|f\|_{p,\alpha} = \sum_{i=1}^m \|X_i f\|_{p,\alpha-1}$

Note $\dim G \le d$ but $d \le D$ need not hold

Trudinger Inequality

Theorem (Saloff-Coste)

G, d, D as above

$1 < p < \infty$, $\alpha p = d$ or $\alpha p = D$

$\hookrightarrow$ $\exists\, c > 0$ s.t.

$$\int_\Omega \exp\left(\left[f(x)/c\,\|f\|_{p,\alpha}\right]^{\frac{p}{p-1}}\right) dx \leq c\,|\Omega|$$

$\forall f \in C_0^\infty(\Omega)$, $\Omega \subset G$ open.

Also harmonic analysis results on multipliers + Riesz Transforms etc.

IV) Basic Lie Theory

G : a Lie Group

- The maps $(g,h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth

Often $G \cong \mathbb{R}^N$, (at least locally)

$gh = (f_1(g,h), \ldots, f_N(g,h))$

- A measure μ on G is called left-invariant on G provided

  $\mu(gA) = \mu(A) \quad \forall g \in G$ and measurable $A \subset G$

  - $gA = \{ga : a \in A\}$ = left translates of A.

- Right-invariant measure is similarly defined

- Every Lie group G has a canonical left-invariant measure called the left Haar measure μ_L satisfying

  - is a Borel measure
  - is left invariant
  - $K \subset G$ cpt $\to \mu_L(K) < \infty$
  - outer regular : $\mu_L(S) = \inf\{\mu(U) : S \subset U, U \text{ open}\}$
  - inner regular : $\mu_L(U) = \sup\{\mu(K) : K \subset U, K \text{ cpt}\}$

- The right Haar measure μ_R is similarly defined.

- μ_L and μ_R are unique up to scaling

- If $\mu_R = \mu_L$, then G is called unimodular

$\mathfrak{g}$: a (finite dimensional real) Lie algebra

- $\mathfrak{g} \cong \mathbb{R}^N$ and $\mathfrak{g}$ has an alternating bilinear map

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \,;\, (X, Y) \mapsto [X, Y]$$

(called a Lie bracket) which satisfies

- $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
  $[X, aY + bZ] = a[X, Y] + b[X, Z]$
- $[X, X] = 0$ $([X, Y] = -[Y, X])$
- Jacobi identity:
  $$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$$

Exercise: $(\mathfrak{g}, [\,,\,])$ is associative iff it is 2-step nilpotent or abelian

Example

$sl(n, \mathbb{R})$ = $n \times n$ real matrices A w/ $\operatorname{tr} A = 0$

and $[A, B] = AB - BA$, $A, B \in sl(n, \mathbb{R})$

Vector Fields:

On $\mathbb{R}^N$: $X = \sum_{j=1}^{N} a_j(x)\, \partial x_j$

First order differential operator.

$\mathfrak{X}(\mathbb{R}^N)$ = Vector fields on $\mathbb{R}^N$

↳ commutator is a Lie bracket:

$[X, Y]u = X(Yu) - Y(Xu)$

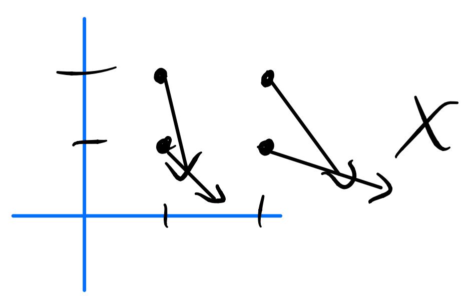
$X, Y \in \mathfrak{X}(\mathbb{R}^N)$, $u \in C^2(\mathbb{R}^N)$

↳ $(\mathfrak{X}(\mathbb{R}^N), [\ ,\ ])$ is a Lie algebra.

Example: $X = x\partial_x - y\partial_y$ on $\mathbb{R}^2$

$X \rightsquigarrow (x, -y)$

$\rightsquigarrow$



On G: $X : G \to TG$, $g \mapsto v \in T_g G$

$\hookrightarrow$ locally

$$X = \sum_{j=1}^{\dim G} a_j(x)\, \partial x_j$$

$\mathfrak{X}(G)$ = vector fields on G - also a Lie algebra w/ $[\ ,\ ]$ = Commutator

- There are special v.f.'s which respect the group structure on G.

- Let $L_g : G \to G$ denote left multiplication:
$$L_g(h) = gh.$$

- A vector field X is called left-invariant if
$$(Xu)(g\,\cdot) = X(u(g\,\cdot)) \qquad \forall g \in G$$
$$(Xu) \circ L_g = X(u \circ L_g)$$

- Let $\mathfrak{X}_L(G)$ = space of left-invariant vector fields.

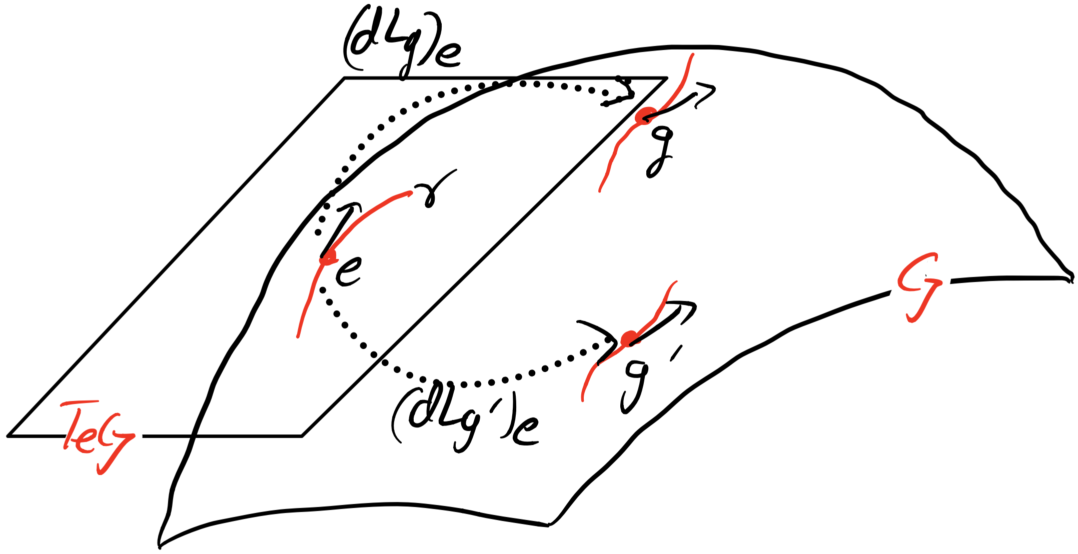
- Also a Lie algebra w/ $[\ ,\ ]$ = commutator

- Let $\mathfrak{g} \triangleq T_e G$
  - $\hookrightarrow v \in \mathfrak{g} \rightsquigarrow X_g = (dL_g)_e v \in T_g G$
  - $\hookrightarrow X \in \mathfrak{X}_L(G),\ X : G \to TG$

$X_g u = (dL_g)_e v(u) = \frac{d}{dt}\Big|_{t=0} u \circ L_g \circ \gamma(t)$

$\gamma(0) = e,\ \gamma'(0) = v \in T_e G$

- $v \mapsto X$ is a Lie Algebra isomorphism
  $\mathfrak{g} \longrightarrow \mathfrak{X}_L(G)$
- $\mathfrak{g}$ is given the bracket
  $[v, w] \triangleq [X_v, X_w]_e$
  $v \mapsto X_v,\ w \mapsto X_w$



* Short hand: $\mathfrak{g} = \operatorname{Lie} G \to \mathfrak{g}$ is the Lie algebra of G.

Key point: We may study $\mathfrak{g}$ or $\mathfrak{X}_L(G)$

Algebraic basis: A collection $a_1, \dots, a_{d'} \in \mathfrak{g}$ s.t. $\exists r \in \mathbb{N}$ s.t. $a_1, \dots, a_{d'}$ and their brackets

$$[a_{i_1}, [a_{i_2}, \dots [a_{i_{n-1}}, a_{i_n}]]], \quad i_j \in \{1, \dots, d'\}, \quad n \leq r$$

span $\mathfrak{g}$. The smallest r is called the **rank** of $\mathfrak{g}$ with respect to $a_1, \dots, a_{d'}$

- In general: $\mathfrak{g}$ may have multiple algebraic bases with different ranks.

Linear basis: usual linear algebra basis.

- $a_1, \dots, a_{d'}$ = algebraic basis, write $A_1, \dots, A_{d'}$ = corresponding left-invariant vector fields.

$(A_d)_g \hat{=} d\, L_g|_e\, a_d$.

- We may call $A_1, \dots, A_{d'}$ or $a_1, \dots, a_{d'}$ an algebraic basis.

Example: $H = \mathbb{R}^3$ with group law
$(x,y,t)(a,b,s) = (x+a,\, y+b,\, t+s+\frac{1}{2}(ya - xb))$
has left invariant vector fields

$$X = \partial_x + \frac{1}{2} y \partial_t, \quad Y = \partial_y - \frac{1}{2} x \partial_t, \quad T = \partial_t$$

which satisfy $[X, Y] = T$.

In particular, $X|_{(x,y,t)=e}$, $Y|_{(x,y,t)=e} \in T_e H$ form an algebraic basis.

H = Heisenberg Group $\sim$ is a GoPG

Example: on $\mathbb{R}^2$: $X = \partial_x$, $Y = x\partial_y$
$[X, Y] = \partial_y$
$\therefore$ $X, Y, [X,Y]$ span $\mathbb{R}^2$ at each point
↳ X, Y not invariant under a Lie group product.

Sub-Laplacian:

On $\mathbb{R}^N$: $X_j = \partial x_j \rightsquigarrow$ Laplacian $= \Delta = \sum_{j=1}^N X_j^2$

On G: $A_1, \dots, A_{d'} \rightsquigarrow$ sub-Laplacian $= \mathcal{L} = \sum_{j=1}^{d'} A_j^2$

Note: $\mathcal{L}$ is not canonical in general

$\mathcal{L}$ is a 2nd order left-invariant differential operator

$$\mathcal{L}(u \circ L_g) = (\mathcal{L} u) \circ L_g \quad \forall g \in G$$

- Play analoguous role as Δ
- Common PDEs to study:

$$\mathcal{L} = 0 \quad \text{(harmonic functions)}$$
$$\mathcal{L} = f \quad \text{(Poisson's Equation)}$$
$$\mathcal{L} = \partial_t \quad \text{(Heat Equation)}$$
$$\mathcal{L} = i\partial_t \quad \text{(Schrödinger's Equation)}$$

- More generally:

On $\mathbb{R}^N$: consider operators

$$L = \sum_{i,j=1}^N a_{ij} \partial_{x_j} \partial_{x_i}, \quad a_{ij} \in \mathbb{R}$$
$$A = (a_{ij}), \quad A \geq \mu I$$

On G: $$H = -\sum_{j=1} c_{k\ell} A_k A_\ell, \quad c_{k\ell} \in \mathbb{C}$$
$$\mathrm{Re}\,(c_{k\ell})_{k\ell} \geq \mu I, \quad \mu > 0.$$

$\hookrightarrow$ H a $\mathbb{C}$-coefficient 2nd order left-invar. 2nd order operator.

V) Geometry

Length Metrics:

$\mathcal{L}$ = a family of paths in $\mathbb{R}^N$ & $\forall x, y \in \mathbb{R}^N$
$\exists\ \mathcal{L} \ni \gamma : [0,1] \to \mathbb{R}^N$ s.t. $\gamma(0) = x$, $\gamma(1) = y$
& one may only travel along these paths

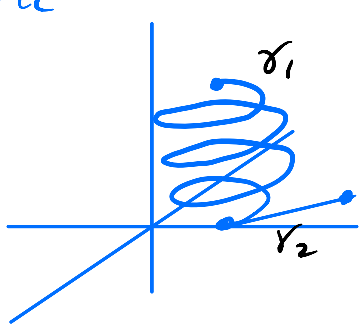
Then
$$d_{\mathcal{L}}(x,y) \triangleq \inf_{\substack{\gamma \in \mathcal{L} \\ \gamma(0)=x \\ \gamma(1)=y}} \int_0^1 |\gamma'(t)|\, dt$$

= "length of shortest path connecting x to y"

measures distance according to this constraint.

↳ Specifying $\mathcal{L}$ introduces a constrained non-Euclidean geometry on $\mathbb{R}^N$

Example



Vector field Defined Length Metric:

G = Lie group, $\mathfrak{g}$ = Lie G

$a_1, \dots, a_d$ = Linear basis for $\mathfrak{g}$

$\exists\ \gamma : [0,1] \to G$, $\gamma(0) = g$, $\gamma(1) = h$, abs. cont. and γ_K s.t.

$$\gamma'(t) = \sum_{K=1}^{d} \gamma_K(t) A_K|_{\gamma(t)} \quad (a.e.)$$

The velocity vectors of γ are linear combinations of $\{A_K|_{\gamma(t)}\}$ for a.e. $t \in [0,1]$.

↳ Define a distance

$$d(g,h) = \inf_{\substack{\gamma(0)=g \\ \gamma(1)=h}} \int_0^1 \left(\sum_{K=1}^{d} \gamma_K^2(t)\right)^{1/2} dt$$

= "length of shortest path $h \mapsto g$"

Elliptic Modulus: $|g| \triangleq d(g,e)$

- d is left-invariant
- All elliptic moduli on G are equivalent:

$$c^{-1}|\cdot|_a \le |\cdot|_b \le c|\cdot|_a$$

(change of basis arguement)

- Global and local geometry independent of linear basis.

Now consider $a_1, \dots, a_{d'}$ = algebraic basis
↳ can still define a distance (despite $d' \leq \dim \mathfrak{g}$)

Why? $A_1, \dots, A_{d'}$ satisfy Hörmander's condition (due to left translation):
$A_1|_g, \dots, A_{d'}|_g$ and their commutator of length $\leq r$ (= their rank) span $T_g G$ $\forall g \in G$.

↳ $\forall g, h \in G$ $\exists \gamma: [0,1] \to G$, $g \mapsto h$, abs. cont. and γ_k s.t.

$$\gamma'(t) = \sum_{k=1}^{d'} \gamma_k(t) A_k|_{\gamma(t)} \quad (a.e.)$$

Note: velocity of γ is constrained to $\operatorname{span}\{A_k : k = 1, \dots, d'\}$

Subelliptic distance:

$$d'(g,h) = \inf_{\substack{\gamma(0)=g \\ \gamma(1)=h}} \int_0^1 \left(\sum_{k=1}^{d'} \gamma_k^2(t)\right)^{1/2} dt$$

a.k.a.: Carnot-Carathéodory metric or Control distance

Subelliptic modulus: $|g|' = d'(g,e)$

If $a_1, \dots, a_{d'} \rightsquigarrow d'(\cdot,\cdot)$, $a_1, \dots, a_{d'}, \dots, a_d \rightsquigarrow d(\cdot,\cdot)$
↳ $d(\cdot,\cdot) \leq d'(\cdot,\cdot)$, $|\cdot| \leq |\cdot|'$
More paths for linear basis

Proposition $|\cdot|'$ = subelliptic modulus for alg. basis of rank r; $|\cdot|$ = elliptic modulus
↳ $\exists c>0: c^{-1}|g| \le |g|' \le c|g|^{1/r}$, $g \in G$ $|g| \le 1$.

rank $r>1 \to |\cdot|$ and $|\cdot|'$ not locally equivalent.

Proposition $|\cdot|, |\cdot|'$ as above.
↳ $\exists \delta>0, c>0: c^{-1}|g| \le |g|' \le c|g|, g \in G, |g| \ge \delta$

All moduli are equivalent outside a compact set.

Local Growth: Fix an algebraic basis $a_1, \dots, a_{d'}$. $|\cdot|', d'(\cdot,\cdot)$ = corresponding modulus and distance.

Haar measure

$B_\rho' = \{g \in G : |g|' < \rho\}$, $V'(\rho) \triangleq |B_\rho'|$ = volume

Proposition $\exists D' \ge d = \dim G$, $C>0, c>0$ s.t.
$c\rho^{D'} \le V'(\rho) \le C\rho^{D'}$, $\rho \in (0,1]$.

D' = local dimension wrt $a_1, \dots, a_{d'}$

. c.s.s.

Thm 3.14

D' depends on basis and but there is a maximal local dimension.

Computing D':

locally Haar measure in terms of Hausdorff meas.

$\mathfrak{g}_1' \triangleq \operatorname{span}\{a_1, \dots, a_{d'}\}$

$\mathfrak{g}_k' \triangleq \operatorname{span}\{a_1, \dots, a_{d'},$ commutators w/ order $\leq k\}$

↳ $\mathfrak{g}_1' \subset \mathfrak{g}_2' \subset \dots \subset \mathfrak{g}_r' = \mathfrak{g}$

Set $\mathfrak{h}_1' = \mathfrak{g}_1'$, choose $\mathfrak{h}_k'$ s.t. $\mathfrak{g}_k' = \mathfrak{h}_k' \oplus \mathfrak{g}_{k-1}'$

↳ $\mathfrak{g} = \mathfrak{h}_1' \oplus \dots \oplus \mathfrak{h}_r'$

↳

$$D' = \sum_{k=1}^{r} k \dim \mathfrak{h}_k'$$
$$d = \sum_{k=1}^{r} \dim \mathfrak{h}_k$$

Global Growth:

Equivalence of $|\cdot|$, $|\cdot|'$ away from $e \in G$

↳ global Growth is intrinsic

↳ Growth of $\rho \mapsto V(\rho)$ independent of basis.

Two Possibilities: Dimension at ∞.

1) $\exists\, D \in \mathbb{N}_{\geq 0}$ and $c, C > 0$ s.t.

$c\rho^D \leq V(\rho) \leq C\rho^D \rightsquigarrow$ Group of Polynomial Growth

2) $\exists\, \lambda, \mu > 0$, $c, C > 0$ s.t.

$ce^{\lambda\rho} \leq V(\rho) \leq Ce^{\mu\rho} \rightsquigarrow$ Group of exponential Growth

Exercise: Suppose G_1, G_2 are connected groups. Suppose G_1 has polynomial growth and G_2 has exponential growth. Show that $G_1 \times G_2$ has exponential growth.

Proposition All nonunimodular groups are of exponential growth

Proof

Δ = modular function

$\Omega \mapsto \mu_L(\Omega g) \rightsquigarrow$ a left-Haar measure

$\hookrightarrow \exists\, \Delta : G \to \mathbb{R}_{>0}$ s.t. $\mu_L(\Omega g) = \Delta(g)\mu_L(\Omega)$

$\exists\, g \in B_1'$ s.t. $\Delta(g) > 1$.

$h \in B_1' \to hg^n \in B_{n+1}' \to B_1' g^n \subset B_{n+1}'$

Use $d(hg^{-1}, e) = d(h, g^{-1}) \leq d(h, e) + d(e, g^{-1})$

$\hookrightarrow \mu_L(B_{n+1}') \geq \mu_L(B_1' g^n) = (\Delta(g))^n \mu_L(B_1')$

$\hookrightarrow$ has exponential growth.

Proposition

G = simply connected and nilpotent

$\dim G = d$

$\mathfrak{g} = \text{Lie } G$, $\{\mathfrak{g}_k\}$ = lower central series

$\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \cdots \oplus \mathfrak{h}_{r_0}$, $\mathfrak{g}_k = \mathfrak{h}_k \oplus \mathfrak{g}_{k+1}$.

$\hookrightarrow$ $D = \sum_{k=1}^{r_0} k \dim \mathfrak{h}_k$ = dim. at infinity.

$D \geq d$ and $D = d$ iff $G = \mathbb{R}^d$.

VI) The Nilshadow and Structure Theory

Assume G is simply connected by passing to its universal covering group.

In general: a GoPG is not even homogeneous (i.e., no dilation structures).

Goal: Suitably approximate a GoPG G by one which is stratified.

Two paths:

① Nilshadow + contractions:

$$G = M \ltimes Q \cong_{\text{mnfld}} G_N = M \times Q_N$$

M = Levi subgrp, cpt, sim. con.,

Q = radical = largest solvable subgrp
a GoPG, sim. con.

Q_N = nilshadow of Q
$\cong_{\text{mnfld}} Q$
a GoPG, nilpotent
grp law = modification of Q's

Now use contractions of N.R.S. to approximate Q_N by a stratified grp.

② Group quotient + Nilshadow:

$G = \tilde{G}/H$, where $\tilde{G}$ is a larger grp whose radical $\tilde{Q}$ has a stratified nilshadow.

$\tilde{G} = M \rtimes \tilde{Q}$

$\tilde{Q}_N$ is stratified

Structure Theory

$\mathfrak{g}$ = real Lie algebra

Ideal: a subsp. $\mathfrak{i} \subset \mathfrak{g}$: $[\mathfrak{g}, \mathfrak{i}] \subset \mathfrak{i}$

Subalgebra: a subsp. $\mathfrak{h} \subset \mathfrak{g}$: $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$

i.e., $\mathfrak{h}$ is a Lie algebra

Solvable: $\mathfrak{g}^{(1)} \triangleq \mathfrak{g}$, $\mathfrak{g}^{(k+1)} \triangleq [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}]$

$\{\mathfrak{g}^{(k)}\} \triangleq$ Derived series.

$\exists k \in \mathbb{N} : \mathfrak{g}^{(k)} = \{0\} \xrightarrow{\Delta} \mathfrak{g}$ is solvable

Radical: the unique solvable ideal $\mathfrak{q} \subset \mathfrak{g}$ containing every other solvable ideal.

Semisimple $\mathfrak{g}$: $\mathfrak{q} = \{0\}$

$[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$

Nilpotent: $\mathfrak{g}_1 \triangleq \mathfrak{g}$, $\mathfrak{g}_{k+1} \triangleq [\mathfrak{g}, \mathfrak{g}_k]$

$\{\mathfrak{g}_k\}$ = lower central series

$\exists r_0 \in \mathbb{N} : \mathfrak{g}_{r_0+1} = \{0\} \neq \mathfrak{g}_{r_0} \xrightarrow{\Delta} \mathfrak{g}$ is nilpotent

Graded: $\mathfrak{g} = \bigoplus_{k \geq 0} \mathfrak{h}_k$ w/ all but finitely many 0 and

$$[\mathfrak{h}_k, \mathfrak{h}_j] \subset \mathfrak{h}_{k+j}.$$

Stratified: $\mathfrak{g}$ graded and $\mathfrak{h}_1$ generates $\mathfrak{g}$; i.e., $\mathfrak{g}$ has an algebraic basis contained in $\mathfrak{h}_1$.

Nilradical: the unique nilpotent ideal $\mathfrak{n} \subset \mathfrak{g}$ containing every other nilpotent ideal.

$\mathfrak{n} \subset \mathfrak{q}$, $\mathfrak{n} = \text{nilradical}(\mathfrak{q})$

Levi subalgebra: a semisimple subalgebra $\mathfrak{m} \subset \mathfrak{g}$: $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{q}$ as v. spaces

- always exist
- Every $\mathfrak{g}$ decomposes into a solvable and semisimple part.

Adjoint representation:

"Ado Thm"

$\mathfrak{g}$ matrix

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$

$$\mathrm{ad}\, X \triangleq [X, \bullet].$$

cs $\mathfrak{q}$ = nilpotent

ad surjective?

$$\mathrm{ad}\,X(Y) = [X, Y]$$

$X, Y \in \mathfrak{g}$

Nilpotent map: $A: \mathfrak{g} \to \mathfrak{g}$. If $\exists\, n \in \mathbb{N}$: $A^n = 0 \overset{\Delta}{\Rightarrow} A$ nilpotent.

Semisimple map: $B: \mathfrak{g} \to \mathfrak{g}$ s.t. each B-invariant subspace V ($BV \subset V$) has a complementary B-invariant subspace W ($\mathfrak{g} = V \oplus W$).

Jordan decomposition:
$\forall a \in \mathfrak{g}\ \exists$ semisimple $S(a): \mathfrak{g} \to \mathfrak{g}$, nilpotent $N(a)$ s.t.

$$\operatorname{ad} a = S(a) + N(a)$$
$$[S(a), N(a)] = 0.$$

Adjoint action decomposes into s.s. and n. parts.

Semidirect product:
$\mathfrak{g}, \mathfrak{h}$ Lie algebras
$\tau: \mathfrak{h} \to \operatorname{End}(\mathfrak{g})$, $\tau[a,b] = [\tau a, \tau b]$
$\hookrightarrow$ $\mathfrak{g} \rtimes_\tau \mathfrak{h} \overset{\Delta}{=} (\mathfrak{g} \oplus \mathfrak{h}, [\ ,\ ]_\tau)$

$$[(a,b),(a',b')]_\tau \overset{\Delta}{=} ([a,a']_{\mathfrak{g}} + \tau(b)a' - \tau(b')a, [b,b']_{\mathfrak{h}})$$

- Levi decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{q} \to \mathfrak{g} = \mathfrak{m} \ltimes \mathfrak{q}$.

On the group level

G a Lie group w/ $\mathfrak{g} = \mathrm{Lie}\, G$.
Call G X iff $\mathfrak{g}$ is X, where
$X \in \{$nilpotent, semisimple, etc$\}$.

Theorem G con., $\mathfrak{h} < \mathfrak{g}$ a subalg.
$\hookrightarrow \exists!$ con. subgrp H w/ $\mathfrak{h} = \mathrm{Lie}(H)$

Theorem $\oint G$ sim. con.
$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{q}$, $\mathfrak{m} \rightsquigarrow M$, $\mathfrak{q} \rightsquigarrow Q$
$\hookrightarrow$ M, Q sim. con. and closed in G,
$M \cap Q = \{e\}$ and

$G = MQ$

$$G = M \rtimes Q.$$

$M \triangleq$ Levi subgrp, $Q =$ radical

$\exists \varphi : Q \longrightarrow \mathrm{Aut}(M)$

$(m, q)(m', q') = (m \varphi(q')m', qq')$

$(m\, q' m' (q')^{-1}, qq')$

The Nilshadow

Nilshadow:

$\mathfrak{g}$ = Lie alg.

$\mathfrak{q} = \mathrm{rad}(\mathfrak{g})$

$\mathfrak{n} = \mathrm{nilrad}(\mathfrak{g})$

$S(a)$ = s.s. part of $\mathrm{ad}\, a$ by Jordan dec.

Fix subsp. $\mathfrak{v} \subset \mathfrak{q}$ s.t.

i) $\mathfrak{q} = \mathfrak{v} \oplus \mathfrak{n}$

ii) $S(\mathfrak{v})\mathfrak{v} = \{0\}$ Such a $\mathfrak{v}$ always exists

Lastly, if $a \in \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{v} \oplus \mathfrak{n}$, write $a_{\mathfrak{v}}$ for the component of a in $\mathfrak{v}$.

Lastly define the bracket

$$[a,b]_N = [a,b] \underbrace{- S(a_{\mathfrak{v}})b + S(b_{\mathfrak{v}})a}_{\text{subtract off the "s.s. part."}}$$

on $\mathfrak{q}$.

The Lie algebra $(\mathfrak{q}, [a,b]_N)$ is called a nilshadow of $\mathfrak{q}$ and denoted by $\mathfrak{q}_N$. It is nilpotent.

∘ $[\ ,\ ]_N$ extends to a bracket

$[\ ,\ ]_N^s : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \Rightarrow \mathfrak{g}_N^s \triangleq (\mathfrak{g}, [\ ,\ ]_N^s)$ = semidirect shadow

$[\ ,\ ]_N^s|_{\mathfrak{m}} = [\ ,\ ]|_{\mathfrak{m}}$, $[\ ,\ ]_N^s|_{\mathfrak{q}} = [\ ,\ ]_N$

$[\ ,\ ]_N^s|_{\mathfrak{m} \times \mathfrak{q}} = [\ ,\ ]|_{\mathfrak{m} \times \mathfrak{q}}$.

Inverting the Shadow

Goal: pass from $\mathfrak{g}_N = \mathfrak{m} \times \mathfrak{q}_N$ and $G_N = M \times Q_N$ back to $\mathfrak{g} = \mathfrak{m} \rtimes \mathfrak{q}$ and $G = M \rtimes Q$.

Shadow:

$\mathfrak{g}_N \triangleq \mathfrak{m} \times \mathfrak{q}_N$ = direct product of Lie algebras $(\mathfrak{m}, [\ ,\ ]_{\mathfrak{g}}|_{\mathfrak{m}}), (\mathfrak{q}_N, [\ ,\ ]_N)$.

Twisted Lie Bracket: $\tau : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ a rep. on $\mathfrak{g}$ s.t. $\tau(\tau(a)b) = 0$.

Then

$$[a, b]_\tau \triangleq [a, b] + \tau(a)b - \tau(b)a$$

defines a Lie bracket on $\mathfrak{g}$.

Write $\mathfrak{g}_\tau = (\mathfrak{g}, [\ ,\ ]_\tau)$.

Theorem Choose $\mathfrak{m}$ and ϑ as before. Define the rep. $\sigma : \mathfrak{g}_N \to \mathrm{End}(\mathfrak{g}_N)$ by

$$\sigma(m, q)(m', q') = (0, (\mathrm{ad}_{\mathfrak{g}} m + S(q_\vartheta))q')$$

$m \in \mathfrak{m}, q \in \mathfrak{q}$.

Then $[\ ,\ ]_\tau = [\ ,\ ]_{\mathfrak{g}}$ and so

$$\mathfrak{g} = \mathfrak{m} \rtimes \mathfrak{q} = (\mathfrak{m} \times \mathfrak{q}_N)_\sigma = (\mathfrak{g}_N)_\sigma$$

- $\mathfrak{g}$ is a twisted Lie algebra of its shadow

∘ σ "adds back" s.s. part which was removed to define $[\,,]_N$ on q_N.

Twisted Products:

$T : G \to \mathrm{Aut}(G)$, $*$ = grp law on G

↳ twisted product

$$g \;{}_T{*}\; h \triangleq (T(h^{-1})g) * h$$

${}_T{*}$ is associative iff $T(T(g)h) = T(h)$

∘ $G_T = (G, {}_T{*})$ is a Lie grp w/

$$e_{G_T} = e_G, \quad h^{(-1)T} = T(h)(h^{-1})$$

${}_T{*}$-inverse

Lie grp shadows:

Let $q = \mathfrak{v} \oplus \mathfrak{n}$ as before.

Let $\tau_g(a) \triangleq -S(a_{\mathfrak{v}})$ so that

$[\;,\;]^s_N = [\;,\;]_\tau$ = bracket on g_N^s

∘ $\tau_g : g \to \mathrm{End}\, g$ descends to a hom.

$$T_G : G \to \mathrm{Aut}(G)$$

$$T_G(\exp a)\exp b = \exp(e^{-S(a_{\mathfrak{v}})} b)$$

$\mathcal{G}_N^s \triangleq (\mathcal{G}, {}_{T_{\mathcal{G}}}*) =$ semidirect shadow

Let $T_Q = T_{\mathcal{G}}|_Q$.

↳ $Q_N = (Q, {}_{T_Q}*) =$ nilshadow

Lastly $\mathcal{G}_N = M \times Q_N$ (direct prod. of grps) is called the shadow of $\mathcal{G}$.

- Just as $\mathfrak{g} \rightsquigarrow \mathfrak{g}_N$ was inverted by

$$\mathfrak{g} = (\mathfrak{g}_N)_\sigma,$$

$\mathcal{G} \rightsquigarrow \mathcal{G}_N$ may be inverted.

- $\sigma : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ descends to a hom.

$$S : \mathcal{G} \to \mathrm{Aut}(\mathcal{G}).$$

Theorem

$$\mathcal{G} = M \rtimes Q = (M \times Q_N, {}_S*)$$

i) Questions

1) G a Lie group with inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g} = \operatorname{Lie} G$

↳ $\langle\langle u, v \rangle\rangle_g = \langle (dL_{g^{-1}})_g u, (dL_{g^{-1}})_g v \rangle$, $u, v \in T_g G$

is a left-invariant Riemann metric on G.

Q: G a GoPG $\Leftrightarrow$ $\langle\langle , \rangle\rangle$ a metric with polynomial growth?

$\Delta_{\langle\langle , \rangle\rangle}$ relates to sub-Laplacians?

Guess: X = algebraic basis w/ sub-Laplacian $\mathcal{L}$ - Declare X orthonormal $\to$ sub-Riemannian inner product $\langle\langle , \rangle\rangle \to \mathcal{L}$ relates to $\mathcal{L}_{\langle\langle , \rangle\rangle}$?

2) What about the Schrödinger equation

$$\mathcal{L} = i\partial_t$$

and the Schrödinger kernel.

G a GoPG $\Rightarrow G = M \rtimes Q$, M cpt, Q GoPG.

$\Rightarrow$ heat kernel behaves like that on $\mathbb{R}^m \times \mathbb{T}^d$.

↳ says things about Schrödinger kernel?

3) What is the maximal local dimension of a given G.

4) Higher order Trudinger inequality?

5) Let $G_1, \dots, G_N$ be GoPG with dimensions at infinity $D_1, \dots, D_N$. Q: is the dimension at infinity of $G_1 \times \dots \times G_N$ $D_1 + \dots + D_N$?

ii) Explicit Examples

1) Let G be the universal covering of the group of Euclidean motions on the plane

↳ G is 3d, solvable and $G \circ PG$

↳ Every sub-Laplacian on $G \rightsquigarrow$ 2nd order operator on $\mathbb{R}^3$ w/ periodic coefficients.

- $\mathfrak{g} = \mathcal{L}ie\, G \to \{X_1, X_2, X_3\}$ a basis w/

  $[X_1, X_2] = X_3$, $[X_1, X_3] = -X_2$, $[X_2, X_3] = 0$

- $G \cong \mathbb{R}^2 \rtimes_\tau \mathbb{R}$, $\tau : \mathbb{R} \to GL(\mathbb{R}^2)$,

  $x \mapsto rot_x =$ ccw rot. by x.

2) Example of non-unimodular group:

Group of affine transformations on $\mathbb{R}$:

$G = \{ x \mapsto ax+b : a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R} \} = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right\}$

$d\mu_L = \frac{1}{a^2} da\, db$, $d\mu_R = \frac{1}{|a|} da\, db$.

Note G is solvable.

iii) Bib

Alexopolous

- An application of homogenezation theory to harmonic analysis on solvable Lie groups of polynomial growth. (1993)

Saloff-Coste

- Analyse sur les groupes de Lie à croissance polynômiale. (1990)
- Sur la décroissance des puissances de convolution sur les groupes

Varopoulos

- Analysis on Lie Groups (1988)

Nagel-Ricci-Stein

- Harmonic Analysis and Fundamental Solutions on Nilpotent Lie Groups (1990)