Math 2410 # Practice Exam III **May 3rd, 2019**

Name: ________________________________

Instructions:

- Answer each question to the best of your ability. Show your work or receive no credit.
- All answers must be written clearly. Be sure to erase or cross out any work that you do not want graded. Partial credit can not be awarded unless there is legible work to assess.
- If you require extra space for any answer, you may use the back sides of the exam pages. Please indicate when you have done this so that I do not miss any of your work.

Academic Integrity Agreement

I certify that all work given in this examination is my own and that, to my knowledge, has not been used by anyone besides myself to their personal advantage. Further, I assert that this examination was taken in accordance with the academic integrity policies of the University of Connecticut.

Signed:____
(full name)

Questions:	1	2	3	4	5	EC	**Total**
Score:							

1. (5 points) Solve the following initial value problem using the Laplace transform:

$$\begin{aligned} y'' + y &= f(t) \\ y(0) &= 0 \\ y'(0) &= 1 \end{aligned}$$

where

$$f(t) = \begin{cases} 0 & 0 \leq t < \pi \\ 1 & \pi \leq t < 2\pi \\ 0 & t \geq 2\pi \end{cases}$$

2. (5 points) Let $f(t)$ be a 2-periodic function defined by $f(t) = t$ on $0 \leq t < 2$. Compute $\mathscr{L}\{e^t * f(t)\}$.

$$\mathcal{L}\{e^t * f(t)\} = \mathcal{L}\{e^t\}\mathcal{L}\{f(t)\} \longleftarrow \text{Convolution}$$

$$= \frac{1}{s-1}\frac{1}{1-e^{-2s}}\int_0^2 e^{-st}t\,dt \longleftarrow \text{periodic formula}$$

$$= \frac{1}{s-1}\frac{1}{1-e^{-2s}}\left[-\frac{e^{-st}}{s}t\Big|_0^2 + \int_0^2 \frac{e^{-st}}{s}dt\right] \longleftarrow \text{Integration by parts}$$

$$= \boxed{\frac{1}{s-1}\frac{1}{1-e^{-2s}}\left[-\frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2}\right]}$$

3. (5 points) Please compute the following Laplace transforms:

(a) $\mathscr{L}\{t^2 \sin t\}$ $= \frac{d^2}{ds^2} \mathcal{L}\{\sin t\} = \frac{d^2}{ds^2} \frac{1}{s^2+1} = \frac{6s^2-2}{(s^2+1)^3}$

(b) $\mathscr{L}\{\mathscr{U}(t-1)\mathscr{U}(t-2)\}$

Note: $\mathcal{U}(t-1)\mathcal{U}(t-2) = \mathcal{U}(t-2)$ $\rightarrow$ $\mathcal{L}\{\mathcal{U}(t-1)\mathcal{U}(t-2)\} = \mathcal{L}\{\mathcal{U}(t-2)\}$

$= \frac{e^{-2s}}{s}$

(c) $\mathscr{L}\left\{\int_0^t \sin(\tau) d\tau\right\}$ $= \frac{\mathcal{L}\{\sin(t)\}}{s} = \frac{1}{s(s^2+1)}$

4. (5 points) Indicate whether the following statements are true or false.

(a) $\mathscr{L}\{af(t)+bg(t)\} = a\mathscr{L}\{f(t)\} + b\mathscr{L}\{g(t)\}$ for all constants a, b, and all functions f, g with Laplace transforms.

T: just the linearity of $\mathcal{L}$.

(b) The function $f(t) = e^{e^t}$ has a Laplace transform.

F: $\lim_{t\to\infty} e^{-st}e^{e^t} = +\infty \to \int_0^\infty e^{-st}e^{e^t}\,dt = \infty$

(c) There is a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and constant vector K so that $X(t) = Kt^{1/2}$ is a solution to $X'(t) = AX(t)$.

F: solution vectors take the form

$X(t) = Ke^{\lambda t}$

where K an evec, λ an eval.

or $X(t) = K_0e^{\lambda t} + K_1te^{\lambda t}$.

(d) There is a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

so that

$$X(t) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\lambda t}$$

is the general solution to $X'(t) = AX(t)$.

T: take $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. Then eval of A is λ.

$\to X(t) = K_0e^{\lambda t} + K_1te^{\lambda t}$ $\quad\to X(t) = K_0e^{\lambda t}$

$K_1 = (A - \lambda I)K_0 = 0$. $\quad = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}e^{\lambda t}$.

5. (5 points) Solve the following matrix DE IVP

$$X'(t) = \begin{pmatrix} 13 & -1 \\ 6 & 8 \end{pmatrix} X(t)$$
$$X(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

det $A = 110$

$\lambda = \dfrac{21 \pm \sqrt{21^2 - 440}}{2} \to \lambda_1 = 10, \lambda_2 = 11$

evecs:

λ_1: $\begin{matrix} (a-\lambda_1)x_1 + bx_2 = 0 \\ cx_1 + (d-\lambda_1)x_2 = 0 \end{matrix} \to \begin{matrix} 3x_1 - x_2 = 0 \\ 6x_1 - 2x_2 = 0 \end{matrix}$

$\to x_1 = \frac{1}{3}x_2$

$\to K_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

λ_2: $\begin{matrix} 2x_1 - x_2 = 0 \\ 6x_1 - 3x_2 = 0 \end{matrix} \to x_1 = \frac{1}{2}x_2$

$\to K_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\therefore$ gen. sol. is $X(t) = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{10t} + c_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{11t}$

IVP: $X(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^0 + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^0$

$= \begin{pmatrix} c_1 + c_2 \\ 3c_1 + 2c_2 \end{pmatrix}$

$\to \begin{matrix} 2 = c_1 + c_2 \\ 1 = 3c_1 + 2c_2 \end{matrix} \to \begin{matrix} 2 = c_1 + c_2 \\ -3 = c_1 \end{matrix} \to c_2 = 5$

$\therefore$ Solution is:

$$\boxed{X(t) = -3\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{10t} + 5\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{11t}}$$

6. (5 points) Solve the following matrix DE IVP

$$X'(t) = \begin{pmatrix} 24 & -5 \\ 45 & -6 \end{pmatrix} X(t)$$
$$X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\det A = 81$

$\lambda = \dfrac{18 \pm \sqrt{18^2 - 4 \cdot 81}}{2}$

$= 9$

$\therefore$ Repeated eval $\rightarrow$ $X(t) = K_0 e^{9t} + K_1 t e^{9t}$

$K_0 = \text{initial value} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$K_1 = (A - \lambda I) K_0 = \left(\begin{pmatrix} 24 & -5 \\ 45 & -6 \end{pmatrix} - \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 15 & -5 \\ 45 & -15 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 30 \end{pmatrix}$

$\therefore \boxed{X(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{9t} + \begin{pmatrix} 10 \\ 30 \end{pmatrix} t e^{9t}}$

The Laplace Transform

Properties

Laplace Transform:	$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$
Linearity:	$\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathscr{L}\{f(t)\} + \beta\mathscr{L}\{g(t)\}$
	$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathscr{L}^{-1}\{F(s)\} + \beta\mathscr{L}^{-1}\{G(s)\}$
Laplace of Derivatives:	$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$
	$\mathscr{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$
	$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
Inverse Laplace of Derivatives:	$\mathscr{L}\{tf(t)\} = -\frac{d}{ds}\mathscr{L}\{f(t)\}$
	$\mathscr{L}\{t^2f(t)\} = \frac{d^2}{ds^2}\mathscr{L}\{f(t)\}$
	$\mathscr{L}\{t^nf(t)\} = (-1)^n\frac{d^n}{ds^n}F(s)$
Translation in s:	$\mathscr{L}\{e^{at}f(t)\} = F(s-a)$
	$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}f(t)$
Translation in t:	$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s)$
	$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$
	$\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}$
Convolution:	$f * g = \int_0^t f(\tau)g(t-\tau)d\tau$
Laplace of Convolution:	$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(s)G(s)$
Inverse Laplace of Product:	$\mathscr{L}^{-1}\{F(s)G(s)\} = f * g$
Laplace of Integral:	$\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$
Laplace of T-Periodic Function:	$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$

Heaviside Function

Heaviside Function: $\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} \iff f(t) = g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

$$f(t) = \begin{cases} 0, & 0 \le t < a \\ g(t), & a \le t < b \\ 0, & t \ge b \end{cases} \iff f(t) = g(t)[\mathscr{U}(t-a) - \mathscr{U}(t-b)]$$

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \frac{e^{-as}}{s}$$

Common Laplace Transforms

$$\mathscr{L}\{1\} = \frac{1}{s} \qquad \mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \mathscr{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2} \qquad \mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}$$

$$\mathscr{L}\{\sinh kt\} = \frac{k}{s^2-k^2} \qquad \mathscr{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$$

$$1 = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} \qquad t^n = \mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} \qquad e^{at} = \mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\}$$

$$\sin kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} \qquad \cos kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$$

$$\sinh kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} \qquad \cosh kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$$

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \frac{e^{-as}}{s}$$

Misc.

$$\lambda = \frac{a+d \pm \sqrt{(a+d)^2 - 4\det A}}{2}$$

$$u(x,t) = \frac{1}{2}[g(x+t)+g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(s)ds$$

$$X(t) = K_0 e^{\lambda t} + K_1 t e^{\lambda t}$$

$$K_1 = (A - \lambda I)K_0$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$