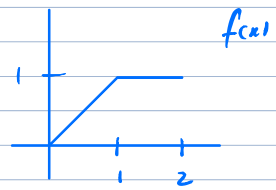
Section 6.4 The Definite Integral

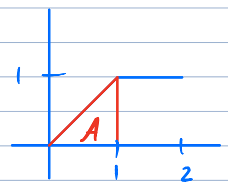
Example

1) Consider



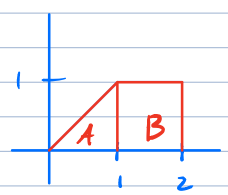
What is the area under the graph on the interval

a) $[0,1]$: Want to find area of red region:



Area $= \frac{1}{2}(\text{base})\times(\text{height}) = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$. We write $\int_0^1 f(x)\,dx = \frac{1}{2}$.

b) $[0,2]$: Want to find area of red region:



Area of A: $\frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$

Area of B: $1\cdot 1 = 1$

Area under graph: $A + B = \frac{1}{2} + 1 = \frac{3}{2}$. We write $\int_0^2 f(x)\,dx = \frac{3}{2}$.

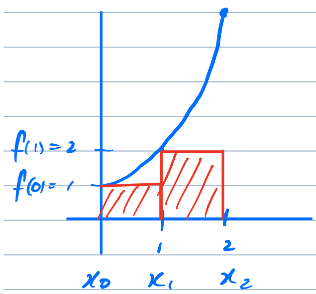
Question: What if graph is curvy?

Example

1) Find area of graph of $f(x) = x^2+1$ over $[0,2]$.

Idea: approximate area with rectangles:

Left hand sum:

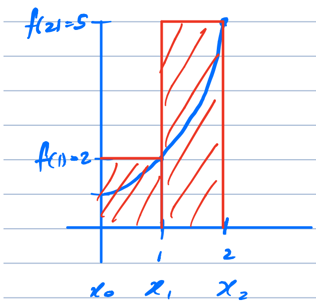


Height of rectangles given by value of f at left end point.

$$\text{Area of rectangles} = (x_1 - x_0) f(x_0) + (x_2 - x_1) f(x_1)$$
$$= 1 \cdot 1 + 1 \cdot 2$$
$$= 3$$

Thus area is ≈ 3

Right hand sum:



Height of rectangles given by value of f at right end point.

$$\text{Area of rectangle} = (x_1 - x_0) f(x_1) + (x_2 - x_1) f(x_2)$$
$$= 1 \cdot 2 + 1 \cdot 5$$
$$= 7$$

Thus area is ≈ 7.

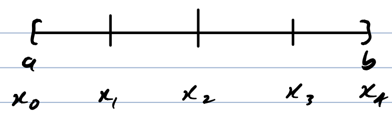
Conclusion: area under $f(x) = x^2+1$ over $[0,2]$ is between 3 and 7.

Riemann sums:

f continuous on $[a,b]$.

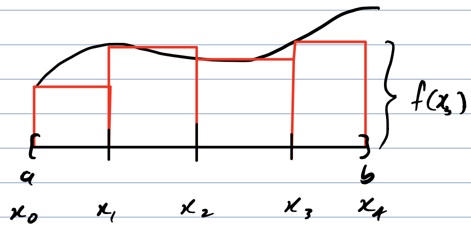
Partition interval into n subintervals of length $\Delta x = \frac{b-a}{n}$

e.g., $n=4$:



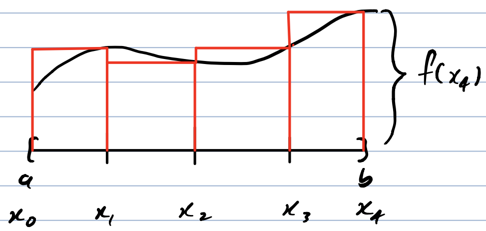
Left hand Riemann sum $= f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$

$$= \sum_{k=0}^{n-1} f(x_k)\Delta x$$



Right hand Riemann sum $= f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$

$$= \sum_{k=1}^{n} f(x_k)\Delta x$$



The definite integral over $[a,b]$:

$$\int_a^b f(x)dx = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to+\infty} \sum_{k=0}^{n-1} f(x_k)\Delta x .$$

If $f \geq 0$, then $\int_a^b f(x)dx$ = area under graph of $f(x)$.

Problems

1) Use left and Right sums with $n=5$ to approximate the integral

$$\int_2^{22} 5x^3+1 \, dx .$$

1. Find Δx:

Given $n=5$, $a=2$, $b=22$

↳ $\Delta x = \frac{b-a}{n} = \frac{22-2}{5} = 4$.

2. Find $x_0, x_1, \ldots, x_n$:

$x_0 = a = 2$, $x_1 = 2+4 = 6$, $x_2 = 6+4 = 10$, $x_3 = 10+4 = 14$, $x_4 = 14+4 = 18$

$x_5 = 18+4 = 22 = b$.

3. Compute sums:

Left sum $= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$

$= f(2)\cdot 4 + f(6)\cdot 4 + f(10)\cdot 4 + f(14)\cdot 4 + f(18)\cdot 4$

$= 41\cdot 4 + 1081\cdot 4 + 5001\cdot 4 + 13721\cdot 4 + 29161\cdot 4$

$= 196{,}020$

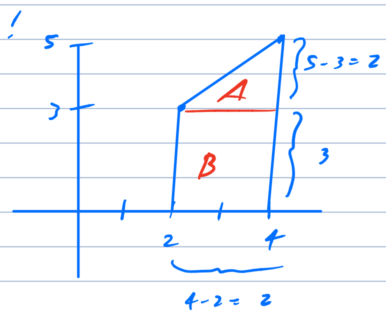
Right sum $= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x$

$= f(6)\cdot 4 + f(10)\cdot 4 + f(14)\cdot 4 + f(18)\cdot 4 + f(22)\cdot 4$

$= 408{,}820.$

2) Compute $\int_2^4 x+1\,dx$ by finding area of an appropriate geometric region.

graph!



$\int_2^4 x+1\,dx =$ area A + area B

$= \frac{1}{2}\cdot 2\cdot 2 + 2\cdot 3$

$= 2 + 6$

$= 8$