Section 5.2 The Second Derivative

Notation:

$$f'' = (f')'$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = y''.$$

<u>Example</u>

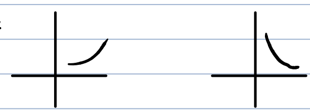
1) if $f(x) = x^2 \ln(8x)$, find $f''(x)$

$$f'(x) = 2x\ln(8x) + x^2 \cdot \frac{8}{8x} = 2x\ln 8x + x$$

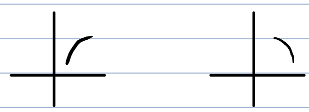
$$f''(x) = \frac{d}{dx}(2x\ln 8x + x) = 2\ln 8x + 2x \cdot \frac{8}{8x} + 1 = 2\ln 8x + 3.$$

Properties of the 2nd Derivative:

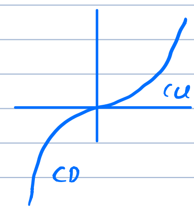
Concave up (CU):

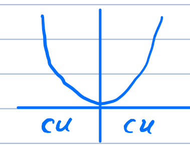


Concave down (CD):



<u>Examples</u>

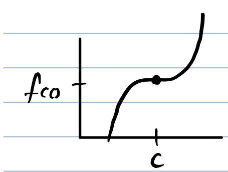




f concave up $\longleftrightarrow$ $f''(x) > 0$ $\longleftrightarrow$ $f'(x)$ increasing

f concave down $\longleftrightarrow$ $f''(x) < 0$ $\longleftrightarrow$ $f'(x)$ decreasing

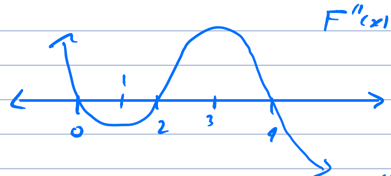
Inflection point: where f changes concavity:



infl. pt: $(c, f(c))$

<u>Problem</u>

1)



a) F is concave up: when $F'' > 0 \Rightarrow (-\infty, 0) \cup (2, 4)$

b) F is concave down: when $F'' < 0 \Rightarrow (0, 2) \cup (4, +\infty)$

c) x-coord. for infl. pts: when F'' changes sign $\Rightarrow x = 0, 2, 4.$

2nd Derivative Test:

$f'(c)=0,\ f''(c)>0 \Rightarrow f(c)$ a rel. min

$f'(c)=0,\ f''(c)<0 \Rightarrow f(c)$ a rel. max

$f'(c)=0,\ f''(c)=0 \Rightarrow$ can't use 2nd Der. Test

<u>Problem</u>

1) $f(x) = x^2 e^{-x}$.

Find where f is incr., decr., CU, CD, infl. pts, rel. ext.

1. $f'(x) = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2-x)$

$f'(x)=0 \rightarrow xe^{-x}(2-x)=0 \rightarrow x=0$ or $e^{-x}=0$ or $2-x=0$

$\rightarrow x=0$ DNE $x=2$

	$(-\infty,0)$	0	$(0,2)$	2	$(2,+\infty)$
test pt	-1		1		3
$f'(x)$	$-$		$+$		$-$
$f(x)$	$\searrow$		$\nearrow$		$\searrow$

$f'(-1) = -3e < 0$
$f'(1) = e^{-1} > 0$
$f'(3) = -3e^{-3} < 0$

incr.: $(0,2)$
decr.: $(-\infty,0) \cup (2,+\infty)$
rel. min.: $f(0) = 0$
rel. max.: $f(2) = 4e^{-2}$
$\Big\} \Rightarrow$ by 1st Der. test.

2. $f''(x) = \frac{d}{dx}\, xe^{-x}(2-x) = e^{-x}(x^2-4x+2)$

$f''(x)=0 \rightarrow e^{-x}(x^2-4x+2)=0 \rightarrow e^{-x}=0$ or $x^2-4x+2=0$

DNE $x = 2 \pm \sqrt{2}$

	$(-\infty,2-\sqrt{2})$	$2-\sqrt{2}$	$(2-\sqrt{2},2+\sqrt{2})$	$2+\sqrt{2}$	$(2+\sqrt{2},+\infty)$
test pt	0		2		5
$f''(x)$	$+$		$-$		$+$
$f(x)$	$\cup$		$\cap$		$\cup$

$f''(0) = 2 > 0$
$f''(2) = -2e^{-2} < 0$
$f''(5) = 7e^{-5} > 0$

CU: $(-\infty, 2-\sqrt{2}) \cup (2+\sqrt{2}, +\infty)$
CD: $(2-\sqrt{2}, 2+\sqrt{2})$
infl. pt at $x = 2-\sqrt{2},\ 2+\sqrt{2}$

2) Find rel. ext. of $f(x) = x^2 \ln x$ using 2nd Der. test.

1. $f'(x) = 2x\ln x + x = x(2\ln x + 1) = 0$

↳ $x = 0$ or $2\ln x + 1 = 0$
But $f(0)$ DNE $\quad \ln x = -\frac{1}{2}$
↳ exclude $\quad x = e^{-\frac{1}{2}}$

2. $f''(x) = 2\ln x + 1 + 2 = 2\ln x + 3$

$f''(e^{-\frac{1}{2}}) = 2\ln e^{-\frac{1}{2}} + 3 = -\frac{1}{2}\cdot 2 + 3 = 2 > 0$

3. $f''(e^{-\frac{1}{2}}) > 0 \Rightarrow f(e^{-\frac{1}{2}}) = -\frac{1}{2}e^{-1}$ = rel. min.

2) $g(x)$ has domain = all real x but $x = 6$.

$g''(x) = \dfrac{(x-3)(x+1)}{(x-6)^2}$.

Find concavity and x-coord of infl. pts. of $g(x)$.

$g''(x) = 0$ when $x = 3, x = -1$
$g''(x)$ DNE when $x = 6$.

	-2	-1	0	3	4	6	7
$g''(x)$	$-$		$+$		$-$		$+$
$g(x)$	$\cap$		$\cup$		$\cap$		$\cup$

$g''(-2) < 0$
$g''(0) > 0$
$g''(4) < 0$
$g''(7) > 0$

$g(x)$ CU on $(-1, 3) \cup (6, +\infty)$
$g(x)$ CD on $(-\infty, -1) \cup (3, 6)$

x-coord of infl. pts. : $x = -1, 3$. Note $g''(6)$ DNE $\Rightarrow$ not infl. pt.