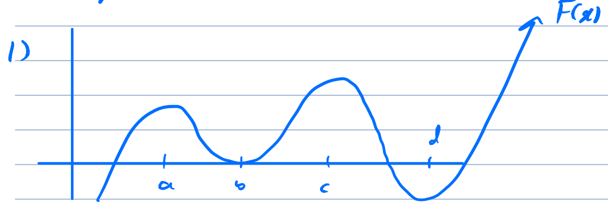
Section 5.1 The First Derivative

Critical Values of $f(x)$: x in domain of f such that $f'(x)=0$ or $f'(x)$ DNE

Example

1)

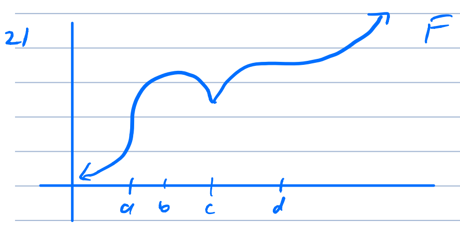


Critical Values:

$F'(x)=0$ when? $x=a,b,c,d$

$F'(x)$ DNE when? no such x values

2)



Critical values:

$F'(x)=0$ when? $x=b,d$

$F'(x)$ DNE when? $x=a,c$.

3) Find Critical values of

a) $f(x)=3x^4+7x^2-8$

$f'(x)=12x^3+14x$

$f'(x)=0$:

$12x^3+14x = x(12x^2+14)=0 \rightarrow x=0$ or $12x^2+14=0$

But $12x^2+14=0 \rightarrow 12x^2=-14 \rightarrow$ no solutions

Thus only $x=0$ is such that $f'(x)=0$.

$f'(x)$ DNE:

f' always exists

CV's: $x=0$.

b) $f(x) = \dfrac{x^4}{x+18}$

$f'(x) = \dfrac{(x+18)4x^3 - x^4}{(x+18)^2}$

$= \dfrac{3x^3(x+24)}{(x+18)^2}$

$f'(x)=0$:

$\dfrac{3x^3(x+24)}{(x+18)^2} = 0 \rightarrow 3x^2(x+24)=0$

$\rightarrow 3x^2 = 0$ or $x+24=0$

$\rightarrow x=0$ or $x=-24$

$f'(x)$ DNE:

$f'(-18)$ DNE. But $f(-18)$ DNE $\rightarrow$ $x=-18$ not in domain $\rightarrow$ not a CV.

CV's $x = 0, -24$.

Properties of First Derivative :

$f'(x) > 0$ on $(a,b) \rightarrow f$ increasing on (a,b)
$f'(x) < 0$ on $(a,b) \rightarrow f$ decreasing on (a,b)

Example
If $f'(x) = \dfrac{4(x-2)}{(x-4)^3}$, find where f is increasing and decreasing

1) Find where $f'(x)=0$ and $f'(x)$ DNE :

$f'(x)=0$:

$\dfrac{4(x-2)}{(x-4)^3} = 0 \rightarrow 4(x-2)=0 \rightarrow x=2$

$f'(x)$ DNE :

$f(4)$ DNE

2) Number line: $(-\infty,2)$ 2 $(2,4)$ 4 $(4,\infty)$

3) Find sign of $f'(x)$ on each interval :

$(-\infty,2)$: Try $x=0$: $f'(0) = \dfrac{4(0-2)}{(0-4)^3} = \dfrac{1}{8} > 0$

$\hookrightarrow f'(x) > 0$ on $(-\infty,2) \rightarrow f \nearrow$ on $(-\infty,2)$

$(2,4)$: Try $x=3$: $f'(3) = \frac{4(3-2)}{(3-4)^3} = -4 < 0$

↳ $f'(x) < 0$ on $(2,4) \to f \searrow$ on $(2,4)$

$(4,+\infty)$: Try $x=5$: $f'(5) = \frac{4(5-2)}{(5-4)^3} = 12 > 0$

↳ $f'(x) > 0$ on $(4,+\infty) \to f \nearrow$ on $(4,+\infty)$.

4) Thus $f(x)$ is increasing on $(-\infty,2) \cup (4,+\infty)$
decreasing on $(4,+\infty)$.

<u>Example</u>

Find where $f(x) = x^4+1$ is increasing and decreasing.

1. $f'(x) = 4x^3$.

$f'(x) = 0$: $4x^3 = 0 \Rightarrow x = 0$

$f'(x)$ DNE: $f'(x)$ always exists.

2.

	$(-\infty,0)$, test -1	0	$(0,+\infty)$, test 1
$f'(x)$	$-$		$+$
$f(x)$	$\searrow$		$\nearrow$

$f'(-1) = 4\cdot(-1)^3 = -4 < 0$

$f'(1) = 4\cdot 1^3 = 4 > 0$

Thus f increasing on $(-\infty,0)$
decreasing on $(0,+\infty)$

<u>Example</u>

If $f'(x) = (x+1)^6(x-6)^5 e^{x-10}$, find where f is increasing and decreasing.

1. CV's: $f'(x) = 0$ when $x=-1$, $x=6$. (Note $e^{x-10} \neq 0$).

2.

	test -2	-1	test 0	6	test 7
$f'(x)$	$-$		$-$		$+$
$f(x)$	$\searrow$		$\searrow$		$\nearrow$

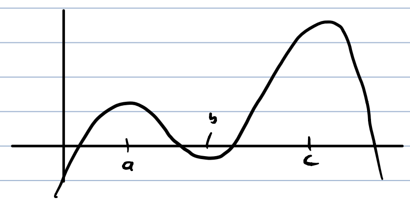
$f'(-2) = (-2+1)^6(-2-6)^5 e^{-2-10}$
$= -32768 e^{-12}$
< 0

$f'(0) = (0+1)^6(0-6)^5 e^{0-10}$
$= -7776 e^{-10}$
< 0

$f(10) = (7+1)^6(7-6)^5 e^{7-10}$
$= 262144 e^{3}$
> 0

Thus
increasing: $(6,+\infty)$
decreasing: $(-\infty,-1) \cup (-1,0)$

Relative extrema:



We call $f(a)$, $f(c)$ relative maximum
$f(b)$ relative minimum

First Derivative Test:

Suppose $f'(c)=0$. Then

	c			
$f'(x)$	$+$	$-$	$\Rightarrow$	$f(c)$ = relative maximum.
$f(x)$	$\nearrow$	$\searrow$		

	c			
$f'(x)$	$-$	$+$	$\Rightarrow$	$f(c)$ = relative minimum
$f(x)$	$\searrow$	$\nearrow$		

	c			
$f'(x)$	$-$	$+$	$\Rightarrow$	$f(c)$ = neither
$f(x)$	$\nearrow$	$\nearrow$		
or	$\searrow$	$\searrow$		

Problems

Find Relative extrema of

1) $f(x) = x^2$

1. $f'(x) = 2x$
   $f'(x) = 0$ when $x = 0$
   $f'(x)$ always exists

2.

	$(-\infty, 0)$	0	$(0, +\infty)$
	-1		$+1$
$f'(x)$	$-$		$+$
$f(x)$	$\searrow$		$\nearrow$

$f'(-1) = -2 < 0$
$f'(1) = 2 > 0$

Thus f has local min. of $f(0)=0$ at $x=0$.

2) $f(x) = x^2 e^{-x^2}$

1. $f'(x) = 2xe^{x^2} - 2x^3 e^{-x^2}$

$f'(x) = 0$:

$$2xe^{x^2} - 2x^3 e^{x^2} = e^{x^2}(2x - 2x^3)$$
$$= 2xe^{x^2}(1 - x^2)$$
$$= 0$$

$\hookrightarrow$

$2x = 0$ or $e^{x^2} = 0$ or $1 - x^2 = 0$

$\hookrightarrow$

$x = 0$ not possible $x = \pm 1$

$\hookrightarrow$ $f'(x) = 0$ when $x = 0, -1, +1$.

$f'(x)$ DNE : $f'(x)$ always exists.

2.

	-2	$-\frac{1}{2}$	$\frac{1}{2}$	2
	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, +\infty)$
$f'(x)$	$+$	$-$	$+$	$-$
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$

$f'(-2) = 12e^{4} > 0$

$f'(-\frac{1}{2}) = -\frac{3}{4} e^{\frac{1}{4}} < 0$

$f'(\frac{1}{2}) = \frac{3}{4} e^{\frac{1}{4}} > 0$

$f'(2) = -12e^{4} < 0$

$x = -1, 0, 1$ in domain of f $\Rightarrow$

Relative min : $f(0) = 0$

Relative max : $f(-1) = e^{-1}$ and $f(1) = e^{-1}$.

3) $f(x) = \frac{1}{x^2}$

1. $f'(x) = -\frac{2}{x^3}$

$f'(x) = 0$: $-\frac{2}{x^3} \neq 0$

$f'(x)$ DNE : $f'(0)$ DNE (Note $f(0)$ DNE)

2.

	-1	1
	$(-\infty, 0)$	$(0, +\infty)$
$f'(x)$	$+$	$-$
$f(x)$	$\nearrow$	$\searrow$

$f'(-1) = 2 > 0$

$f'(1) = 2 < 0$

Note $f(0)$ is not rel. max since $f(0)$ not defined.