Section 3.1 Limits

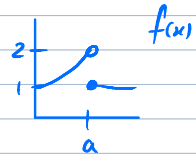
Notation:

$x \to a^-$: "go to a from left"

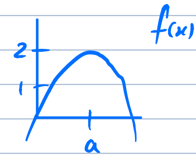
$x \to a^+$: "go to a from right"

Example

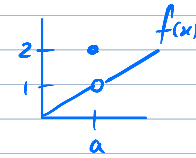
a)



b)



c)

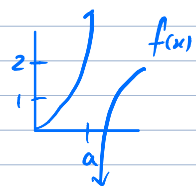


a) $x \to a^-$: $f(x) \to 2$; $x \to a^+$: $f(x) \to 1$

b) $x \to a^-$: $f(x) \to 2$; $x \to a^+$: $f(x) \to 2$

c) $x \to a^-$: $f(x) \to 1$; $x \to a^+$: $f(x) \to 1$

d)



$x \to a^-$: $f(x) \to +\infty$

$x \to a^+$: $f(x) \to -\infty$

If, as $x \to a^-$, $f(x) \to b$, write

$$\lim_{x \to a^-} f(x) = b = \text{"Left hand limit"}$$

If, as $x \to a^+$, $f(x) \to b$, write

$$\lim_{x \to a^+} f(x) = b = \text{"Right hand limit"}$$

If

$$\lim_{x \to a^-} f(x) = b = \lim_{x \to a^+} f(x)$$

write

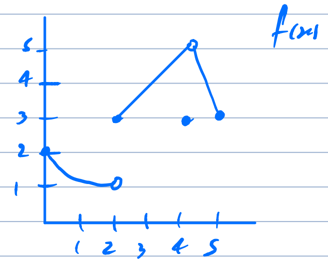
$$\lim_{x \to a} f(x) = b = \text{"limit of } f(x) \text{ as } x \to a\text{"}$$

If

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+}$$

write $\lim_{x \to a} f(x)$ DNE.

Example

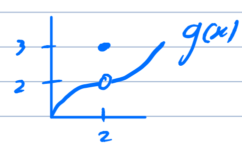


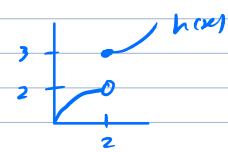
$\lim_{x\to 2^-} f(x) = 1$ $\qquad \lim_{x\to 2^+} f(x) = 3$ $\qquad \lim_{x\to 2} f(x)$ DNE

$\lim_{x\to 4^-} f(x) = 5$ $\qquad \lim_{x\to 4^+} f(x) = 5$ $\qquad \lim_{x\to 4} f(x) = 5$, $\quad f(4) = 3$

Continuity:

If $\lim_{x\to a} f(x) = f(a)$, say f is continuous at a.





$\lim_{x\to 2} f(x) = 2 = f(2)$ $\qquad \lim_{x\to 2} g(x) = 3 \neq g(2) = 2$ $\qquad \lim_{x\to 2} h(x)$ DNE

f continuous at $x=2$ $\qquad$ g discontinuous at $x=2$ $\qquad$ h discontinuous at $x=2$

Continuity = "no jumps or holes"

Examples:

polynomials, exponentials, rational functions, logarithms are continuous where defined.

Problem

1) where is $\frac{x+1}{x^2+3x+2}$ continuous?

Cont. where defined:

$$\frac{x+1}{x^2+3x+2} = \frac{x+1}{(x+1)(x+2)} = \frac{1}{x+2} \quad \text{for } x \neq -1, -2$$

Thus domain is $(-\infty,-2) \cup (-2,-1) \cup (-1,+\infty)$

Trick:

To compute
$$\lim_{x\to a} f(x)$$

Try $f(a)$:

if $f(a) = b$, then $\lim_{x\to a} f(x) = b$

if $f(a) = \frac{\#}{0}, \pm\infty$ then limit DNE

if $f(a) = \frac{0}{0}, \frac{\pm\infty}{\pm\infty}$, then need to do more work.

Problems

1) $\lim_{x\to 0} \frac{x-32}{x-4}$

Try $\frac{0-32}{0-4} = \frac{-32}{-4} = 8 \to \lim_{x\to 0} \frac{x-32}{x-4} = 8$

2) $\lim_{x\to 3} \frac{x^2-5x+6}{x-3}$

Try $\frac{3^2-5\cdot 3+6}{3-3} = \frac{0}{0} \to$ need more work.

Try factoring:
$$\frac{x^2-5x+6}{x-3} = \frac{(x-3)(x-2)}{x-3} = x-2 \quad \text{for } x \neq 3.$$

Thus $\lim_{x\to 3} \frac{x^2-5x+16}{x-3} = \lim_{x\to 3} (x-2) = 3-2 = 1.$

Tip: if quadratic, try factoring

3) $\lim_{x\to 16} \frac{\sqrt{x}-4}{x-16}$

Try $\frac{\sqrt{16}-4}{16-16} = \frac{0}{0} \to$ need more work

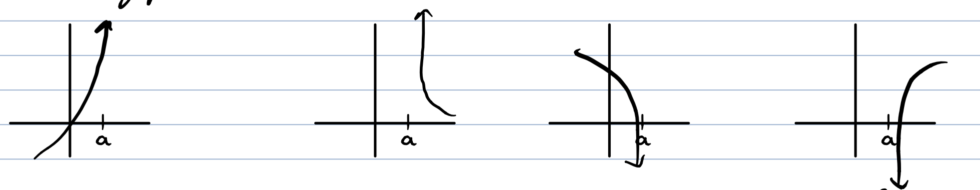
Try rationalizing:
$$\frac{\sqrt{x}-4}{x-16} \cdot \frac{\sqrt{x}+4}{\sqrt{x}+4} = \frac{x-16}{(x-16)(\sqrt{x}+4)} = \frac{1}{\sqrt{x}+4} \quad \text{for } x \neq 16$$

Thus
$$\lim_{x\to 16} \frac{\sqrt{x}-4}{x-16} = \lim_{x\to 16} \frac{1}{\sqrt{x}+4} = \frac{1}{\sqrt{16}+4} = \frac{1}{8}$$

Tip: if radical, try rationalizing

4) $\lim\limits_{h\to 0} \dfrac{h^2}{h+h^2} = \lim\limits_{h\to 0} \dfrac{h}{h}\,\dfrac{h}{1+h} = \lim\limits_{h\to 0} \dfrac{h}{1+h} = \dfrac{0}{1+0} = 0$

Vertical Asymptotes:



$\lim\limits_{x\to a^-} f(x) = +\infty$ $\quad \lim\limits_{x\to a^+} f(x) = +\infty$ $\quad \lim\limits_{x\to a^-} f(x) = -\infty$ $\quad \lim\limits_{x\to a^+} f(x) = -\infty$

Tip: functions always discontinuous at vertical asymptotes

Examples

i) $\lim\limits_{x\to 0^-} \dfrac{1}{x} = \text{"}\dfrac{1}{0^-}\text{"} = -\infty$

$\lim\limits_{x\to 0^+} \dfrac{1}{x} = \text{"}\dfrac{1}{0^+}\text{"} = +\infty$

Tip: 0^- = "negative 0"
0^+ = "positive 0"

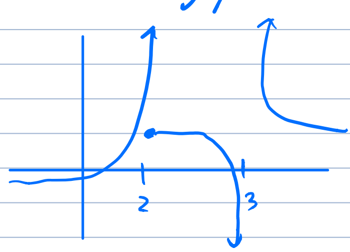
ii) $\lim\limits_{x\to 0^-} \dfrac{1}{x^2} = \text{"}\dfrac{1}{(0^-)^2}\text{"} = \text{"}\dfrac{1}{0^+}\text{"} = +\infty$

$\lim\limits_{x\to 0^+} \dfrac{1}{x^2} = \text{"}\dfrac{1}{(0^+)^2}\text{"} = \text{"}\dfrac{1}{0^+}\text{"} = +\infty$

iii) $\lim\limits_{x\to 1^-} \dfrac{x}{x^2-x} = \lim\limits_{x\to 1^-} \dfrac{x}{x}\,\dfrac{1}{x-1} = \lim\limits_{x\to 1^-} \dfrac{1}{x-1} = \text{"}\dfrac{1}{1^- - 1}\text{"} = \text{"}\dfrac{1}{0^-}\text{"} = -\infty$

Tip: a^+ = "barely greater than a" $\to$ $a^+ - a = 0^+$
a^- = "barely less than a" $\quad a^- - a = 0^-$

iv) Find Vertical Asymptotes:



at $x = 2, 3$.

Limit of Piecewise Function

Problem

1) For
$$f(x) = \begin{cases} x+1 & x \geq 2 \\ x-1 & 1.9 \leq x < 2 \\ x^3 & x < 1.9 \end{cases}$$

Compute $\lim_{x \to 2^-} f(x)$:

2^- = "just left of 2" $\to$ we use rule for $x < 2$ and close to 2

$\to$ use rule for $1.9 \leq x < 2$

$\to \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x-1) = 2-1 = 1.$