7.3 Operational Properties

Given an ODE like

$$a_n y^{(n)} + \cdots + a_0 y = f(t)$$

recall $f(t)$ is called a driving function.

Think: $f(t)$ – input signal to system
sol. y – response.

Q: What happens to y if we modify $f(t)$? (e.g., $f(t) \to e^{at} f(t)$)

$\therefore$ If we want to use $\mathcal{L}$, need to know how $\mathcal{L}$ behaves under modification.

7.3.1 Translation on the s-axis

Theorem (First translation Theorem)

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a) \quad \text{or} \quad \mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$$

Proof:

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt = \int_0^\infty e^{-st+at} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt = F(s-a)$$

$\square$

Ex

a) $\mathcal{L}\{e^{at} t^n\} = ?$

Recall $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} = F(s) \to \mathcal{L}\{e^{at} t^n\} = F(s-a) = \frac{n!}{(s-a)^{n+1}}$

b) $\mathcal{L}\{e^{at} \cos kt\} = ?$

Recall $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2} = F(s) \to \mathcal{L}\{e^{at} \cos kt\} = F(s-a) = \frac{s-a}{(s-a)^2+k^2}$

c) $\mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = ?$

Partial fractions: $\frac{2s+5}{(s-3)^2} = \frac{2}{s-3} + \frac{11}{(s-3)^2}$

Recall: $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$\therefore \mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} = 2e^{3t} + 11e^{3t}t.$

Ex

Solve

$$\begin{cases} y'' - 6y' + 9y = t^2 e^{3t} \\ y(0) = 2 \\ y'(0) = 17 \end{cases}$$

1) $\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{t^2 e^{3t}\}$

2) $s^2Y - sy(0) - y'(0) - 6sY + 6y(0) + 9Y = \frac{2}{(s-3)^3}$ (Recall $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$)

$s^2Y - 2s - 17 - 6sY + 12 + 9Y = \frac{2}{(s-3)^3}$

$(s^2 - 6s + 9)Y - 2s - 5 = \frac{2}{(s-3)^3}$

$Y = \frac{2}{s-3} + \frac{11}{(s-3)^2} + \frac{2}{(s-3)^5}$ (Partial Fractions)

$\therefore \quad y = \mathcal{L}^{-1}\{Y\} = 2\mathcal{L}^{-1}\{\frac{1}{s-3}\} + 11\mathcal{L}^{-1}\{\frac{1}{(s-3)^2}\} + \frac{2}{4!}\mathcal{L}^{-1}\{\frac{4!}{(s-3)^5}\}$

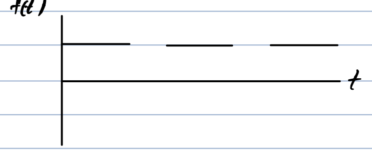
$= 2e^{3t} + 11e^{3t}t + \frac{2}{4!}e^{3t}t^4$

7.3.2 Translation on the t-Axis

Given

$$a_n y^{(n)} + \cdots + a_0 y = f(t),$$

the input $f(t)$ might be "on-off":



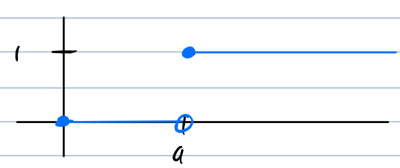
How do we handle such f?

Definition: The function

$$\mathcal{U}(t-a) = \begin{cases} 0 & 0 \le t < a \\ 1 & t \ge a \end{cases}$$

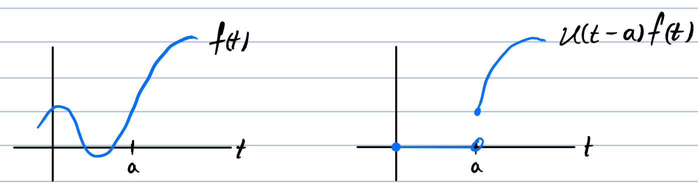
is called the unit step function or Heaviside function.

Graph of $\mathcal{U}(t-a)$:



Remarks

i) $\mathcal{U}(t-a)$ "cuts off" functions to left of a:



ii) $\mathcal{U}(t-a)$ allows us to write piecewise functions algebraically:

a) $f(t)=\begin{cases} g(t) & 0\le t<a \\ h(t) & t\ge a\end{cases} \longleftrightarrow f(t)=g(t)-g(t)\,\mathcal{U}(t-a)+h(t)\,\mathcal{U}(t-a)$

$0\le t<a \to \mathcal{U}(t-a)=0 \to f(t)=g(t)-g(t)\cdot 0+h(t)\cdot 0=g(t)$
$t\ge a \to \mathcal{U}(t-a)=1 \to f(t)=g(t)-g(t)\cdot 1+h(t)\cdot 1=h(t)$

b) $f(t)=\begin{cases} 0 & 0\le t<a \\ g(t) & a\le t<b \\ 0 & t\ge b\end{cases} \longleftrightarrow f(t)=g(t)[\mathcal{U}(t-a)-\mathcal{U}(t-b)]$

$0\le t<a \to f(t)=g(t)[0-0]=0$
$a\le t<b \to f(t)=g(t)[1-0]=g(t)$
$t\ge b \to f(t)=g(t)[1-1]=0$

Ex
$f(t)=\begin{cases} 20t & 0\le t<5 \\ 0 & t\ge 5\end{cases} \to f(t)=20t-20t\,\mathcal{U}(t-5)+0\,\mathcal{U}(t-5)$
$=20t-20t\,\mathcal{U}(t-5)$

Theorem (Second Translation Theorem)
If $F(s)=\mathcal{L}\{f(t)\}$ and $a>0$, then

$$\mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\}=e^{-as}F(s) \longleftrightarrow f(t-a)\,\mathcal{U}(t-a)=\mathcal{L}^{-1}\{e^{-as}F(s)\}$$

and

$$\mathcal{L}\{f(t)\,\mathcal{U}(t-a)\}=e^{-as}\mathcal{L}\{f(t+a)\}$$

Ex
a) $\mathcal{L}\{\mathcal{U}(t-a)\}=?$

$f(t)=1 \to F(s)=\frac{1}{s}$ $\therefore \mathcal{L}\{\mathcal{U}(t-a)\}=e^{-as}\frac{1}{s}$

b) $\mathcal{L}\{(t-2)^2\,\mathcal{U}(t-2)\}=?$

$a=2,\ f(t-2)=(t-2)^2 \to f(t)=t^2 \to F(s)=\frac{2}{s^3}$ $\therefore \mathcal{L}\{(t-2)^2\,\mathcal{U}(t-2)\}=e^{-2s}\frac{2}{s^3}$

c) $\mathcal{L}\{e^t\,\mathcal{U}(t-3)\}=?$

$a=3,\ f(t)=e^t \to \mathcal{L}\{e^t\,\mathcal{U}(t-3)\}=e^{-3s}\mathcal{L}\{e^{t+3}\}=e^{-3s}e^3\mathcal{L}\{e^t\}=e^{-3(s-1)}\frac{1}{s-1}$

Ex
Let $f(t)=\begin{cases} t & 0\le t<3 \\ e^{t-3} & t\ge 3\end{cases}$. Find $\mathcal{L}\{f(t)\}$

Recall:
$$f(t)=g(t)-g(t)\,\mathcal{U}(t-a)+h(t)\,\mathcal{U}(t-a).$$

Here $g(t)=t,\ h(t)=e^{t-3},\ a=3.$

$\therefore$ $f(t)=t-t\,\mathcal{U}(t-3)+e^{t-3}\,\mathcal{U}(t-3)$

$$\therefore \mathcal{L}\{f(t)\} = \mathcal{L}\{t\} - \mathcal{L}\{t\,\mathcal{U}(t-3)\} + \mathcal{L}\{e^{t-3}\,\mathcal{U}(t-3)\}$$
$$= \frac{1}{s} - e^{-3s}\mathcal{L}\{t+3\} + e^{-3s}\frac{1}{s-1}$$
$$= \frac{1}{s} - e^{-3s}\mathcal{L}\{t\} - e^{-3s}\mathcal{L}\{3\} + e^{-3s}\frac{1}{s-1}$$
$$= \frac{1}{s} - e^{-3s}\frac{1}{s^2} - e^{-3s}\frac{3}{s} + e^{-3s}\frac{1}{s-1}$$

Ex

a) $\mathcal{L}^{-1}\{\frac{1}{s-4}e^{-2s}\}$

Want to use $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,\mathcal{U}(t-a)$

$a = 2$, $F(s) = \frac{1}{s-4} \to f(t) = \mathcal{L}^{-1}\{\frac{1}{s-4}\} = e^{4t}$

$\therefore \mathcal{L}^{-1}\{e^{-2s}\frac{1}{s-4}\} = f(t-2)\,\mathcal{U}(t-2) = e^{4t-8}\,\mathcal{U}(t-2)$

b) $\mathcal{L}^{-1}\{\frac{s}{s^2+9}e^{-\pi s/2}\}$

Want to use $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,\mathcal{U}(t-a)$

$a = \frac{\pi}{2}$, $F(s) = \frac{s}{s^2+9} \to f(t) = \mathcal{L}^{-1}\{\frac{s}{s^2+9}\} = \cos(3t)$

$\therefore \mathcal{L}^{-1}\{\frac{s}{s^2+9}e^{-\pi s/2}\} = f(t-\frac{\pi}{2})\,\mathcal{U}(t-\frac{\pi}{2}) = \cos(3t-\frac{3\pi}{2})\,\mathcal{U}(t-\frac{\pi}{2})$.