7.2 Inverse Transform and Transform of Derivatives

Goal: solve ODEs using $\mathcal{L}$. $\therefore$ need a way of translating back.

7.2.1 Inverse Transform

Given $F(s)$, if $f(t)$ is such that $\mathcal{L}\{f(t)\} = F(s)$, we write $f(t) = \mathcal{L}^{-1}\{F(s)\}$ and call $f(t)$ the inverse Laplace transform of $F(s)$.

Ex

a) $\mathcal{L}^{-1}\{\frac{1}{s}\} = 1$ (Recall $\mathcal{L}\{1\} = \frac{1}{s}$)

b) $\mathcal{L}^{-1}\{\frac{1}{s+3}\} = e^{-3t}$ (Recall $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$)

Ex

Find the inverse Laplace transform of

a) $F(s) = \frac{1}{s^5}$

Recall $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$.

$\therefore \mathcal{L}\{\frac{1}{n!}t^n\} = \frac{1}{s^{n+1}} \to \mathcal{L}^{-1}\{\frac{1}{s^5}\} = \frac{1}{4!}t^4 = \frac{1}{24}t^4$

b) $F(s) = \frac{1}{s^2+7}$

Recall $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$

$\therefore \mathcal{L}\{\frac{\sin kt}{k}\} = \frac{1}{s^2+k^2} \to \mathcal{L}^{-1}\{\frac{1}{s^2+7}\} = \mathcal{L}^{-1}\{\frac{1}{s^2+(\sqrt{7})^2}\} = \frac{\sin\sqrt{7}t}{\sqrt{7}}$

c) $F(s) = \frac{1}{(s-1)(s-2)}$

Partial fraction: $\frac{1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2} \to 1 = A(s-2) + B(s-1)$

$\to 1 = B,\ -1 = A$

$\therefore \frac{1}{(s-1)(s-2)} = -\frac{1}{s-1} + \frac{1}{s-2}$

$\therefore \mathcal{L}^{-1}\{-\frac{1}{s-1} + \frac{1}{s-2}\} = -\mathcal{L}^{-1}\{\frac{1}{s-1}\} + \mathcal{L}^{-1}\{\frac{1}{s-2}\}$ (see theorem below)

$= -e^{t} + e^{2t}$

Theorem

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}$$

Remarks:

i) $\mathcal{L}^{-1}\{F(s)\}$ need not be uniquely defined. Won't matter for us.

7.2.2. Transformation of Derivative

Observation:

- $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty \frac{d}{dt}(e^{-st}) f(t)\,dt$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)\,dt \qquad (\text{assume } f(t)e^{-st} \to 0)$$

$$= -f(0) + s\mathcal{L}\{f(t)\}$$

$$= -f(0) + sF(s)$$

- $\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt = e^{-st} f'(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f'(t)\,dt$

$$= -f'(0) + s\left(e^{-st} f(t)\Big|_0^\infty + s^2\int_0^\infty e^{-st} f(t)\,dt \qquad (\text{assume } f'e^{-st} \to 0)\right)$$

$$= -f'(0) - s f(0) + s^2 F(s) \qquad (\text{assume } fe^{-st} \to 0)$$

Theorem

Assume f nice enough. Then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

where $F(s) = \mathcal{L}\{f(t)\}$

Now we can solve Linear IVPs: We go by example

Ex

Solve
$$\begin{cases} \frac{dy}{dt} + 3y = 12\sin 2t \\ y(0) = 6 \end{cases}$$

Solution:

1) Apply $\mathcal{L}$ to get equation in terms of $Y = \mathcal{L}\{y\}$
2) Solve for $Y = \mathcal{L}\{y\}$
3) Apply $\mathcal{L}^{-1}$ to get $y = \mathcal{L}^{-1}\{Y(s)\}$

1) $\mathcal{L}\{\frac{dy}{dt} + 3y\} = \mathcal{L}\{12\sin 2t\}$

$\mathcal{L}\{\frac{dy}{dt}\} + 3\mathcal{L}\{y\} = 12\mathcal{L}\{\sin 2t\}$

2) $sY - y(0) + 3Y = 12\,\frac{2}{s^2+4}$

$Y(s+3) - 6 = \frac{24}{s^2+4}$

$\therefore \quad Y = \frac{24}{(s^2+4)(s+3)} + \frac{6}{s+3}$

$= \frac{8}{s+3} - 2\,\frac{s}{s^2+4} + \frac{6}{s^2+4}$ (Partial fraction)

3) $y = \mathcal{L}^{-1}\{Y\} = 8\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathcal{L}\left\{\frac{2}{s^2+4}\right\}$

$= 8e^{-3t} - 2\cos 2t + 3\sin 2t$

Ex

Solve
$$\begin{cases} y'' - 3y' + 2y = e^{-4t} \\ y(0) = 1 \\ y'(0) = 5 \end{cases}$$

1) $\mathcal{L}\{y'' - 3y' + 2y\} = \mathcal{L}\{e^{-4t}\}$

$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-4t}\}$

2) $s^2 Y(s) - s y(0) - y'(0) - 3sY(s) + 3y(0) + 2Y(s) = \frac{1}{s+4}$ (Recall: $\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$, $\mathcal{L}\{y'\} = sY - y(0)$, $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$)

$s^2 Y - s - 5 - 3sY + 3 + 2Y = \frac{1}{s+4}$

$(s^2 - 3s + 2)Y - s - 2 = \frac{1}{s+4}$

$\therefore \quad Y = \frac{1}{s^2 - 3s + 2}\left(\frac{1}{s+4} + s + 2\right)$

$= -\frac{16/5}{s-1} + \frac{25/6}{s-2} + \frac{1/30}{s+4}$

3) $y = \mathcal{L}^{-1}\{Y\} = -\frac{16}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$

$= -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$