The Wave Equation

1) Multivariable Review
2) The Transport Equation
3) Solving the Wave Equation

1) Multivariable Review

- Partial Differentiation

$f = f(x,y)$ = a function of two variables x and y

$\frac{\partial f}{\partial x} = f_x$ = How much $f(x,y)$ changes when x changes

$\frac{\partial f}{\partial y} = f_y$ = How much $f(x,y)$ changes when y changes

- Multivariable Chain Rule

$x = x(t)$ – treat x as a function of t

$y = y(t)$ – treat y as a function of t

Question: If we give t a nudge, how does $f = f(x(t), y(t))$ respond?

Let:

Δt = a change in t

Δx = a change in x

Δy = a change in y

Observation:

$\Delta x = \frac{dx}{dt}\Delta t$

$\Delta y = \frac{dy}{dt}\Delta t$

Conclusion:

Δf = change in f as a response to a change in x and y

$$= \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt}\Delta t + \frac{\partial f}{\partial y}\frac{dy}{dt}\Delta t$$

$$\frac{\Delta f}{\Delta t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\Delta t \longrightarrow 0 \implies \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

2) The Transport Equation

- Physical Setup

  A wave travels with a constant velocity c and a constant profile

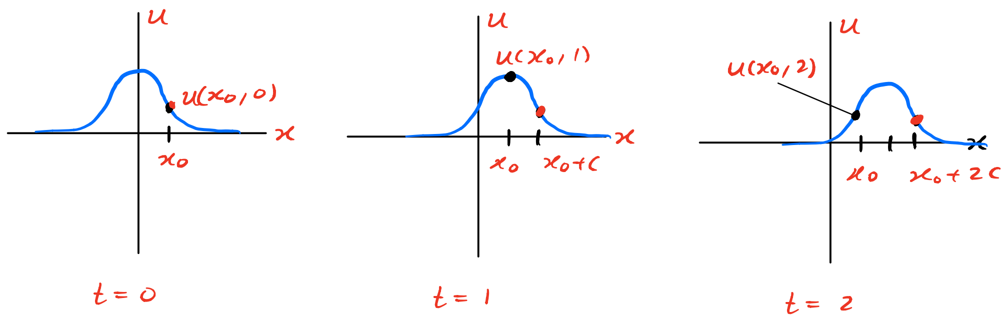
  $u(x,t)$ = Displacement of wave at the position x and time t

- Goal:

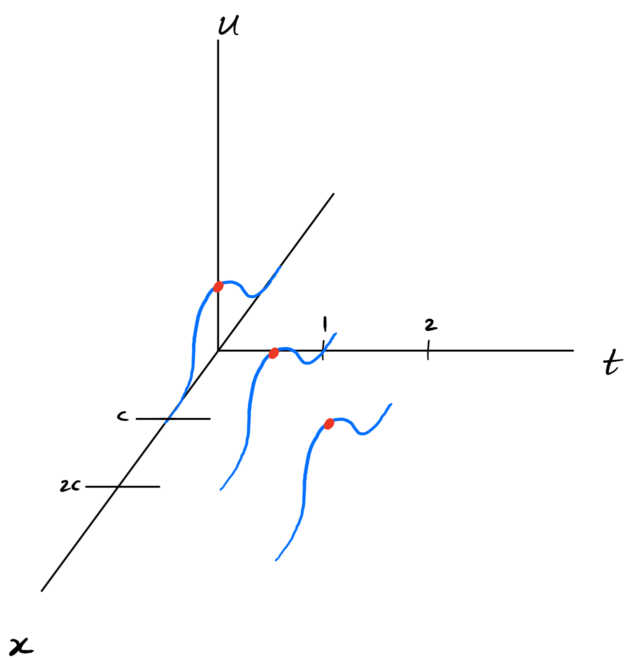
  Setup a PDE which describes such a wave

- Begin with some geometry:

  Time slices:

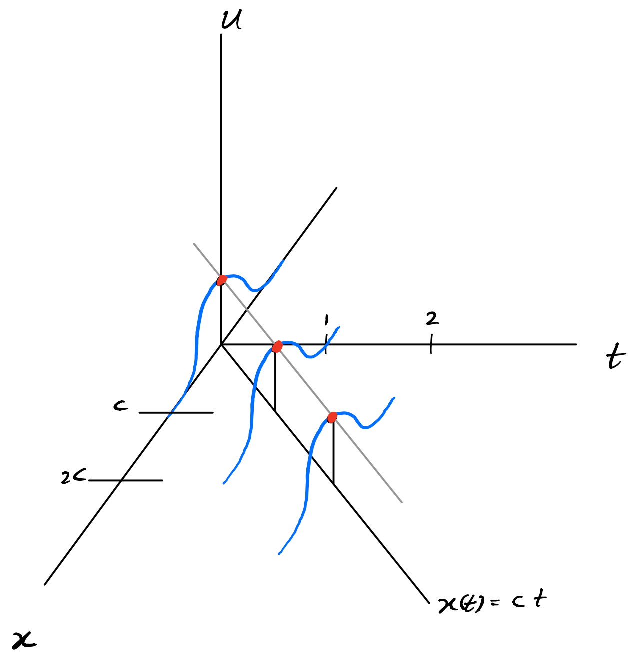


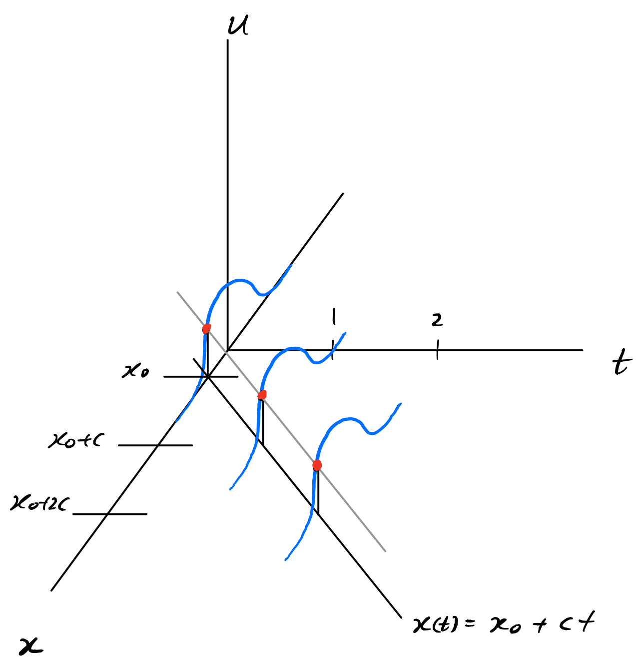
  3D Picture:



∘ Key Observation: $u(x,t)$ is constant on the lines:

$$\boxed{x(t) = x_0 + ct}$$





- Conclusion:
  - $u(x,t)$ is constant on the lines of slope c

    <u>Implies</u>

  - If (x_1,t_1) and (x_2,t_2) lie on the same line of slope c, then:

    $$u(x_1,t_1) = u(x_2,t_2)$$

    <u>But!</u>

  - All points (x,t) lying on the line passing through $(x_0,0)$ of slope c satisfy

    $$x = x_0 + ct$$

    <u>Therefore</u>

    $$\boxed{u(x_0+ct,\, t) = \text{constant}}$$

  Put another way:

  $$u(x_0+ct_1, t_1) = u(x_0+ct_2, t_2), \text{ for all } t_1, t_2$$

  - The Transport Equation

    By the multivariable chain rule:

    $$\begin{aligned}\frac{d}{dt}u(x_0+ct,t) &= u_x(x_0+ct,t)\frac{d}{dt}(x_0+ct) + u_t(x_0+ct,t)\frac{dt}{dt}\\ &= cu_x(x_0+ct,t) + u_t(x_0+ct,t)\\ &= \frac{d}{dt}(\text{constant})\\ &= 0\end{aligned}$$

    But this holds for all x_0 and t!

    Therefore $u(x,t)$ must satisfy the Transport Equation:

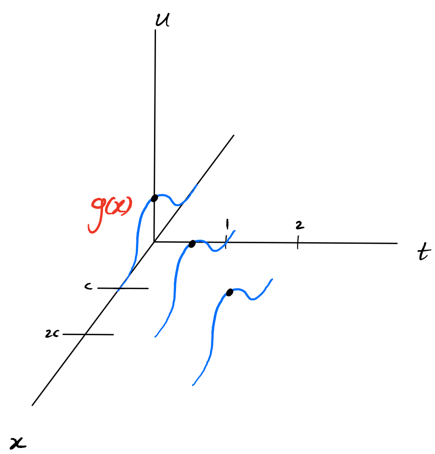
    $$\boxed{u_t + cu_x = 0 \text{ for all } -\infty < x < +\infty,\ -\infty < t < +\infty}$$

∘ The Homogeneous Transport IVP

We wish to solve:

$$\begin{cases} c u_x + u_t = 0 & -\infty < x < +\infty,\ t > 0 \\ u(x,0) = g(x) & -\infty < x < +\infty \end{cases}$$

Initial condition $u(x,0) = g(x)$: initial wave profile



But...

∘ $u(x,t)$ is constant on lines $x(t) = x_0 + ct$!

∘ $u(x_0 + c \cdot 0, 0) = g(x_0)$ for all x_0!

∘ The point $(x_0, 0)$ lies on the line $x(t) = x_0 + ct$

Therefore...

$$u(x_0 + ct, t) = g(x_0) \text{ for } -\infty < x_0 < +\infty,\ t \geq 0$$

In conclusion, after the substitutions $x_0 \longrightarrow x \longrightarrow x - ct$:

$$\boxed{u(x,t) = g(x - ct)}$$

solves the transport equation with initial condition $g(x)$

Remark:

$u(x,t)$ "transports" $g(x)$ from $(x,0)$ to $(x+ct, t)$

- The Inhomogeneous Transport IVP

Now we wish to solve

$$\begin{cases} c u_x + u_t = f(x,t) & -\infty < x < +\infty,\ t > 0 \\ u(x,0) = g(x) & -\infty < x < +\infty \end{cases}$$

where $f(x,t)$ is a specified function

But...

Multivariable Chain rule:

- $\frac{d}{ds} u(x+cs, s) = \frac{d}{ds}(x+cs) u_x(x+cs, s) + \frac{ds}{ds} u_t(x+cs, s)$
$= c u_x(x+cs, s) + u_t(x+cs, s)$
$= f(x+cs, s)$

FTOC:

- $u(x+ct, t) - u(x,0) = \int_0^t \frac{d}{ds} u(x+cs, s) ds = \int_0^t f(x+cs, s) ds$

And...

- $u(x,0) = g(x)$

Therefore, by the substitution $x \longrightarrow x - ct$:

- $u(x,t) = u(x-ct, 0) + \int_0^t f(x + c(s-t), s) ds$
$= g(x-ct) + \int_0^t f(x + c(s-t), s) ds$

Remark:

Now $u(x,t)$ "transports" and evolves $g(x)$:

$$(x,0) \longrightarrow (x+ct, t)$$
$$g(x) \longrightarrow g(x) + \int_0^t f(x+cs, s) ds$$

Compared to homogeneous case $f(x,t) = 0$:

$$(x,0) \longrightarrow (x+ct, t)$$
$$g(x) \longrightarrow g(x)$$

3) The Wave Equation

° In General

$u = u(x_1, x_2, \dots, x_n, t)$ = wave height at position $(x_1, \dots, x_n)$ and time t

The Laplacian:

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

The Wave Equation:

$$\Box u = \frac{\partial^2 u}{\partial t^2} - \Delta u = 0$$

$n=1$: $u_{tt} - u_{xx} = 0$ $\longleftrightarrow$ vibrating string

$n=2$: $u_{tt} - u_{x_1 x_1} - u_{x_2 x_2} = 0$ $\longleftrightarrow$ vibrating membrane

$n=3$: $u_{tt} - u_{x_1 x_1} - u_{x_2 x_2} - u_{x_3 x_3} = 0$ $\longleftrightarrow$ elastic solid

Remarks:

1) $u(x,t)$ reprensents displacement in some direction

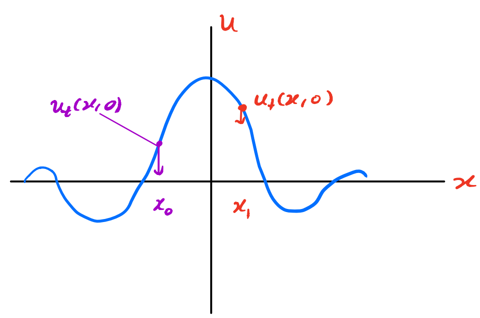
2) $\Box$ is sometimes called the D'Alembertian

3) We will focus on $n=1$. Higher dimensions rely on multivariable calc.

° One dimensional Wave Equation IVP

Now we wish to solve

$$\begin{cases} u_{tt} - u_{xx} = 0 & -\infty < x < +\infty,\ 0 < t \\ u(x,0) = g(x) & -\infty < x < +\infty \\ u_t(x,0) = h(x) & -\infty < x < +\infty \end{cases}$$



$t = 0$

Here:

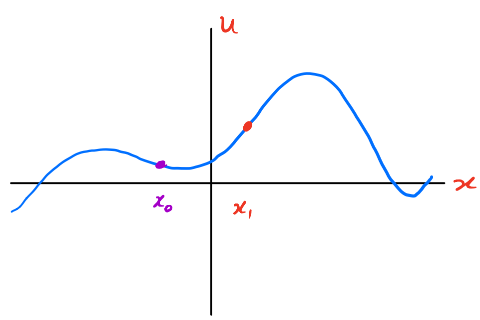
$g(x)$ = initial wave profile

$h(x)$ = initial wave velocity

For example:

Displacement at x_0 might be quick

Displacement at x_1 might be slow



$t = 0.1$

° Solving the Wave Equation

1. Observe: $u_{tt} - u_{xx} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u$

2. Then:

$$u_{tt} - u_{xx} = 0 \longrightarrow \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = 0$$

3. Let

$$v(x,t) = u_t(x,t) - u_x(x,t) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u$$

4. By 1. and 3.:

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u &= \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)(u_t - u_x) \\ &= \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) v \\ &= v_t + v_x\end{aligned}$$

5. By 2. and 4.:

$$u_{tt} - u_{xx} = 0 \longrightarrow \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = 0 \longrightarrow v_t + v_x = 0$$

But $v_t + v_x = 0$ is the transport equation with $c = 1$!

6. By initial conditions of $u(x,t)$ and 3., $v(x,y)$ has initial conditions:

$$\begin{matrix} u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{matrix} \longrightarrow u_x(x,0) = g'(x) \longrightarrow \begin{aligned} v(x,0) &= u_t(x,0) - u_x(x,0) \\ &= h(x) - g'(x) \end{aligned}$$

7. By 5. and 6., $v(x,t)$ solves the homogeneous transport equation:

$$\begin{cases} v_x + v_t = 0 & -\infty < x < +\infty,\ t > 0 \\ v(x,0) = h(x) - g'(x) & -\infty < x < +\infty \end{cases}$$

8. But the solution to this transport equation is:

$$v(x,t) = v(x-t,0) = h(x-t) - g'(x-t)$$

9. Therefore, by 3. and 8., $u(x,t)$ satisfies:

$$u_t - u_x = v(x,t) = h(x-t) - g'(x-t)$$

But $u_t - u_x = h(x-t) - g'(x-t)$ is the nonhomogeneous transport equation with $c = -1$!

10. By initial conditions of $u(x,t)$ and 9., $u(x,t)$ satisfies

$$\begin{cases} -u_x + u_t = h(x-t) - g'(x-t) & -\infty < x < +\infty,\ t > 0 \\ u(x,0) = g(x) & -\infty < x < +\infty \end{cases}$$

11. By 10., $u(x,t)$ is given by:

$$u(x,t) = g(x+t) + \int_0^t h(x-(s-t)-s) - g'(x-(s-t)-s)\, ds$$

$$\boxed{= \tfrac{1}{2}[g(x+t) + g(x-t)] + \tfrac{1}{2}\int_{x-t}^{x+t} h(s)\, ds}$$

∘ Example

$$\begin{cases} u_{tt} - u_{xx} = 0 & -\infty < x < +\infty, \ 0 < t \\ u(x,0) = \sin x & -\infty < x < +\infty \\ u_t(x,0) = \cos x & -\infty < x < +\infty \end{cases}$$

1. We observe the initial conditions:

$$u(x,0) = g(x) = \sin x$$

$$u_t(x,0) = h(x) = \cos x$$

2. By 1. and

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(s)\,ds$$

We conclude

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\sin(x+t) + \sin(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \cos(s)\,ds \\ &= \frac{1}{2}[\sin(x+t) + \sin(x-t)] + \frac{1}{2}[\sin(x+t) - \sin(x-t)] \\ &= \sin(x+t) \end{aligned}$$