5.1 Linear Models: Initial Value Problems

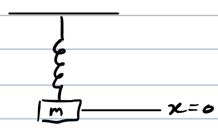
Study

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = g(t)$$
$$y(0) = y_0$$
$$y'(0) = y_1$$

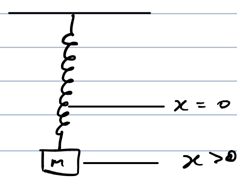
Driving function: $g(t)$
Response: a solution $y(t)$.

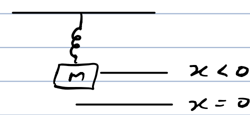
5.1.1 Spring/Mass Systems: Free Undamped Motion

Equilibrium:



In Motion:





DE of Free undamped Motion:

$$m\frac{d^2x}{dt^2} = -kx \longleftrightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0, \quad \omega^2 = k/m$$

m = mass
k = spring constant = $\dfrac{\text{weight of mass}}{\text{how far mass stretches spring}}$

weight = $W = g \cdot m$, $g = \begin{cases} 32\ ft/s^2 \\ 9.8\ m/s^2 \end{cases}$ depends on units used

This equation describes simple harmonic motion / free undamped motion

Free: no driving force
undamped: no resisting force against the motion

Equation of Motion:

$$x'' + \omega^2 x = 0 \rightarrow m^2 + \omega^2 = 0 \rightarrow m = \pm i\omega \rightarrow x(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$$

Period of motion: $T = 2\pi/\omega$ = time (in sec) it takes mass to complete one cycle of motion
Extreme displacement: maximal displacement from equilibrium

Ex

mass of 2 lbs stretches a spring $\frac{1}{2}$ ft. At $t=0$, the mass is released from a point $\frac{2}{3}$ ft below equilibrium position with upward velocity of $\frac{4}{3}$ ft/s. Find equation of motion

Want to solve

$$x'' + \omega^2 x = 0, \quad \omega^2 = k/m$$
$$x(0) = \tfrac{2}{3}$$
$$x'(0) = -\tfrac{4}{3}$$

m: $\quad W = 2 = mg \rightarrow m = \frac{W}{g} = \frac{2}{32} = \frac{1}{16}$ slug

K: $\quad k = \dfrac{\text{weight}}{\text{how far mass stretches spring}} = \dfrac{2}{1/2} = 4$ lb/ft

$\therefore \omega^2 = k/m = \frac{4}{1/16} = 64 \rightarrow \omega = 8$

$\therefore$ equation of motion is

$$x(t) = c_1 \cos 8t + c_2 \sin 8t$$

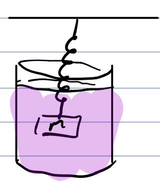
IVP: $x(0) = 2/3$, $x'(0) = -4/3 \rightarrow c_1 = \frac{2}{3}$, $c_2 = -\frac{1}{6}$

$\therefore \quad x(t) = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t$ is the equation of motion

5.1.2 Spring/Mass Systems: Free Damped Motion

Now assume mass experience resisting (damping) force from surrounding medium.

E.g.:



Assume: damping forces are proportional to instant. velocity:

$$F_{damp} = \beta x'.$$

DE of Free Damped Motion:

$$mx'' = -kx - \beta x' \longleftrightarrow x'' + 2\lambda x' + \omega^2 x = 0$$

$\beta > 0$: damping constant
$2\lambda = \beta/m$
$\omega^2 = k/m$

Aux. Eq. $m^2 + 2\lambda m + \omega^2 = 0 \longrightarrow m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2}$, $m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2}$

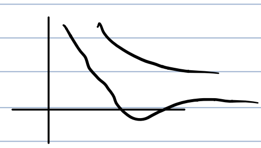
Three Cases for Damping:

Case I: $\lambda^2 - \omega^2 > 0$ ($\beta >> k$)

Overdamped motion: $x(t) = c_1 e^{(-\lambda + \sqrt{\lambda^2-\omega^2})t} + c_2 e^{(-\lambda - \sqrt{\lambda^2 - \omega^2})t}$

$$= \underbrace{e^{-\lambda t}}_{\text{damping}} (\underbrace{c_1 e^{\sqrt{\lambda^2-\omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}}_{\text{exponential behavior}})$$

Eq. describes non-oscillatory motion:

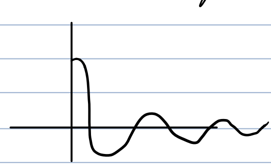


Case III: $\lambda^2 - \omega^2 < 0$ ($\beta << k$)

Underdamped Motion: $m_1 = -\lambda + i\sqrt{\omega^2 - \lambda^2}$, $m_2 = -\lambda - i\sqrt{\omega^2 - \lambda^2}$

$$x(t) = \underbrace{e^{-\lambda t}}_{\text{damping}} (\underbrace{c_1 \cos\sqrt{\omega^2-\lambda^2}\,t + c_2 \sin\sqrt{\omega^2-\lambda^2}\,t}_{\text{oscillation}})$$

Eq. describes decayed oscillation:



Case II: $\lambda^2 - \omega^2 = 0$ ($\frac{\beta^2}{4} = k$)

Critically Damped Motion: $m_1 = m_2 = -\lambda$

$$x(t) = \underbrace{e^{-\lambda t}}_{\text{damping}} (\underbrace{c_1 + c_2 t}_{\text{critical motion}})$$

Critical: $\begin{cases} \text{increase } \beta \longrightarrow \lambda^2-\omega^2>0 \longrightarrow \text{overdamped} \\ \text{decrease } \beta \longrightarrow \lambda^2-\omega^2<0 \longrightarrow \text{underdamped.} \end{cases}$

Ex

mass of 8 lb stretches spring 2 ft. Assume damping force is 2 times instant. velo. mass is released from equilibrium position with an upward velocity of 3 ft/s. Find eq. of motion and extreme displacement.

Want to solve

$$x''+2\lambda x'+\omega^2 x=0$$
$$x(0)=0$$
$$x(0)=-3$$

$$m=\frac{W}{g}=\frac{8}{32}=\frac{1}{4}$$

$$k=\frac{W}{2}=\frac{8}{2}=4$$

$$\beta=2$$

$$2\lambda=\frac{\beta}{m}=\frac{2}{1/4}=8$$

$$\omega^2=\frac{k}{m}=\frac{4}{1/4}=16$$

$$\therefore \quad x''+8x'+16x=0$$

aux. eq. $m^2+8m+16=0 \longrightarrow m_1=m_2=-4$

$$\therefore \quad x(t)=c_1e^{-4t}+c_2te^{-4t}=e^{-4t}(c_1+c_2t)$$

$\therefore$ $x(0)=0$, $x'(0)=-3 \longrightarrow c_1=0, c_2=-3$

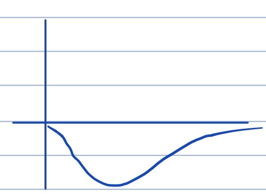
$$\therefore \quad x(t)=-3te^{-4t}$$

Extreme displacement: max or min of $x(t)$:

$$x'(t)=-3e^{-4t}+12te^{-4t}=0 \iff -3+12t=0 \iff t=\frac{1}{4}$$

$$x''(t)=12e^{-4t}+12e^{-4t}-48te^{-4t}$$
$$=24e^{-4t}-48te^{-4t}>0$$

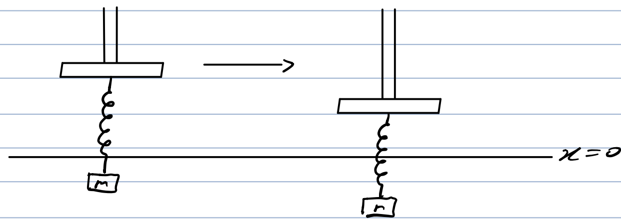
$\therefore$ Extreme disp. is $x(\frac{1}{4})=-\frac{3}{4e}$



5.1.3 Spring / Mass System: Driven Motion

Assumption: something drives the support with force $f(t)$

Driven motion:



- note equilibrium position does not change.

DE of Driven Motion with Damping:

$$mx'' = -kx - \beta x' + f(t) \longleftrightarrow x'' + 2\lambda x + \omega^2 x = F(t)$$

$$2\lambda = \beta/m, \quad \omega^2 = k/m, \quad F(t) = f(t)/m$$

Method of solution: undetermined coefficients or variation of parameters

Recall the gen. sol. takes form

$$x(t) = x_c(t) + x_p(t).$$

Now call:

$x_c(t)$ = transient solution
$x_p(t)$ = steady-state solution

Ex
Interpret and solve the IVP:

$$\frac{1}{5}x'' + 1.2x' + 2x = 5\cos(4t)$$
$$x(0) = \frac{1}{2}$$
$$x'(0) = 0$$

Compare to $mx'' + \beta x' + kx = f(t)$ (we use kg, m, but slugs, ft also work):

mass $m = \frac{1}{5}$ kg
damping $\beta = 1.2$
spring constant $k = 2$
driving force $f(t) = 5\cos(4t)$
mass released $\frac{1}{2}$ m from equilibrium at no initial velocity

Solution: Standard form: $x'' + 6x' + 10x = 25\cos 4t$

1) $m^2 + 6m + 10 = 0 \rightarrow m_1 = -3 + i, \; m_2 = -3 - i$

$\therefore \quad x_c(t) = e^{-3t}(c_1 \cos t + c_2 \sin t)$

2) Guess $x_p(t) = A\cos 4t + B\sin 4t$

$x_p'(t) = -4A\sin 4t + 4B\cos 4t$

$x_p''(t) = -16A\cos 4t - 16B\sin 4t$

plug into $x'' + 6x' + 10x = 25\cos 4t$ and find $A = \frac{-25}{102}$, $B = \frac{50}{51}$

3) $x(t) = \underbrace{x_c(t)}_{\text{transient}} + \underbrace{x_p(t)}_{\text{steady state}}.$

Now use initial condition to find c_1, c_2.

5.1.4 Series Circuit Analogue

Just a renaming of terms

$x \to q$:

$$L q'' + Rq' + \frac{1}{C} q = E(t).$$

$$i(t) = q'(t)$$

Solution $q(t) = \underbrace{q_c(t)}_{\text{transient charge}} + \underbrace{q_p(t)}_{\text{steady-state charge}} \longrightarrow i(t) = \underbrace{i_c(t)}_{\text{transient current}} + \underbrace{i_p(t)}_{\text{steady-state current}}$