4.4 Undetermined Coefficients – Superposition Approach

Preliminary Theory

Consider

(*) $a_n(x)\,y^{(n)} + a_{n-1}(x)\,y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$ (nonhomogeneous)

(**) " " $= 0$ associated homogeneous equation.

Theorem (General Solution – Nonhomogeneous Equations)

- Let $y_c = c_1y_1 + \cdots + c_ny_n$ be the general solution to (**) on I
- Let y_p be **any** particular solution (i.e., free of parameters) to (*) on I

Then the general solution to (*) is

$$y = \underbrace{y_c}_{\text{called complementary function}} + y_p$$

Proof Consider $n=1$ for simplicity:

Let $Y(x)$ and y_p be any two particular solutions to $a_1(x)y' + a_0(x)y = g(x)$.

Let $u(x) = Y(x) - y_p(x)$.

$$\begin{aligned}\therefore\quad a_1(x)u' + a_0(x)u &= a_1(x)(Y - y_p)' + a_0(x)(Y - y_p)\\ &= a_1(x)Y' + a_0(x)Y - a_1(x)y_p' - a_0(x)y_p\\ &= g(x) - g(x)\\ &= 0.\end{aligned}$$

$\therefore$ u is a particular solution to $a_1(x)y' + a_0(x)y = 0$.

$\therefore$ y_c gen sol. to this hom. eq. $\Rightarrow$ $u = c_1y_1 + c_2y_2$ for some c_1, c_2

$\therefore$ $Y - y_p = c_1y_1 + c_2y_2$

$\therefore$ $Y = c_1y_1 + c_2y_2 + y_p$

$\therefore$ every particular solution is obtainable from specifying constants in

$$y = y_c + y_p$$

$\therefore$ $y = y_c + y_p$ is the gen. sol.

$\square$

Theorem (Superposition Principle – Nonhomogeneous Equations)

- y_{p_i} particular solution to $a_n(x)y^{(n)} + \cdots + a_0(x)y = g_i(x)$, $i = 1, \ldots, k$.

Then $y_{p_1} + \cdots + y_{p_k}$ solves $a_n(x)y^{(n)} + \cdots + a_0(x)y = g_1(x) + \cdots + g_k(x)$.

Proof

Consider $n=1$, $k=2$.

- y_{p_i} solves $a_1(x)y' + a_0(x)y = g_i(x)$, $i = 1, 2$

$$\therefore \quad a_1(x)(y_{p_1} + y_{p_2})' + a_0(x)(y_{p_1} + y_{p_2}) = a_1(x)y_{p_1}' + a_0(x)y_{p_1} + a_1(x)y_{p_2}' + a_0(x)y_{p_2}$$
$$= g_1(x) + g_2(x).$$

$\square$

Big Picture

Goal: Solve

$$a_n y^{(n)} + \cdots + a_0 y = g(x).$$

Steps

1) Solve for complementary function y_c to associated hom. eq.
2) Find any particular solution y_p
3) Use gen. sol. theorem to conclude gen. sol. is
$$y = y_c + y_p.$$

Main focus.

Method of Undetermined Coefficients

- $g(x)$ = lin. comb. of products of polynomials, $e^{\alpha x}$, $\cos\alpha x$, $\sin\beta x$
- $\frac{dg}{dx}$ = " ":

$$\text{polynomial} \xrightarrow{\frac{d}{dx}} \text{polynomial}$$
$$\text{cosine or sine} \xrightarrow{\frac{d}{dx}} \text{sine or cosine}$$
$$\text{exponential} \xrightarrow{\frac{d}{dx}} \text{exponential}$$

Call $g(x)$ a polyexpotrig (PET) function.

Thus: derivatives of PET functions are PET functions

Heuristic: $g(x)$ a PET
y_p solves $a_n y^{(n)} + \cdots + a_0 y_p = g(x)$

$\therefore$ guess y_p a PET

$PET = y_p \rightarrow a_n y_p^{(n)} + \cdots + a_0 y_p \rightarrow$ another $PET = g(x)$.

Ex
Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6$

1) Solve $y'' + 4y' - 2y = 0$ for y_c:

$$y_c = C_1 e^{-(2+\sqrt{6})x} + C_2 e^{(2+\sqrt{6})x}$$

2) Guess y_p:

$$g(x) = 2x^2 - 3x + 6 \longrightarrow y_p = Ax^2 + Bx + C$$

$$\therefore \quad 2x^2 - 3x + 6 = (y_p)'' + 4y_p' - 2y_p$$
$$= 2A + 4(2Ax + B) - 2(Ax^2 + Bx + C)$$
$$= -2Ax^2 + (8A - 2B)x + 2A + 4B - 2C$$

$$\therefore \quad 2 = -2A \rightarrow A = -1$$
$$-3 = 8A - 2B = -8 - 2B \rightarrow B = -\tfrac{5}{2}$$
$$6 = 2A + 4B - 2C = -2 - 10 - 2C \rightarrow C = -9$$

$$\therefore \quad y_p = -x^2 - \tfrac{5}{2}x - 9$$

3) gen. sol. is thus

$$y = y_c + y_p = C_1 e^{-(2+\sqrt{6})x} + C_2 e^{(2+\sqrt{6})x} - x^2 - \tfrac{5}{2}x - 9.$$

Ex
Find particular solution to $y'' - y' + y = 2\sin 3x$

We try $y_p = A\sin 3x$:

$$-9A\sin 3x - 3A\cos 3x + \sin 3x = 2\sin 3x$$

only cosine term $\Rightarrow A = 0$

$\therefore$ We add a correction term and try:

$$y_p = A\sin 3x + B\cos 3x$$

$\therefore$

$$-9A\sin 3x - 9B\cos 3x - 3A\cos 3x + 3B\sin 3x + A\sin 3x + B\cos 3x = 2\sin 3x$$

$$(-9A+3B+A)\sin 3x + (-9B-3A+B)\cos 3x = 2\sin 3x$$

$$\therefore \begin{array}{l} -8A+3B = 2 \\ -8B-3A = 0 \end{array} \longrightarrow A = -\tfrac{16}{73},\ B = \tfrac{6}{73}$$

$$\therefore \quad y_p = -\tfrac{16}{73}\sin 3x + \tfrac{6}{73}\cos 3x$$

Case II: a function in y_p is also in y_c (Case I in book)

Ex $y'' - 5y' + 4y = 8e^x \to y_c = c_1 e^x + c_2 e^{4x}$, $y_p = Ae^x$. Both have e^x.

Why is this a problem:

We compute
$$8e^x = (Ae^x)'' - 5(Ae^x)' + 4Ae^x$$
$$= Ae^x - 5Ae^x + 4Ae^x$$
$$= 0$$

Uh Oh!

Multiplication Rule for Case II: if any y_{p_i} in $y_p = y_{p_1} + \cdots + y_{p_n}$ contains a function in y_c, multiply that y_{p_i} by x^n for first n so no duplication happens.

Ex continued

$y_c = c_1 e^x + c_2 e^{4x}$, $y_p = Ae^x$.

both have e^x, so we replace y_p w/ $y_p = Axe^x$:

$$\therefore (Axe^x)'' - 5(Axe^x)' + 4Axe^x = Axe^x + 2Ae^x - 5Axe^x - 5Ae^x + 4Axe^x$$
$$= -3Ae^x$$
$$= 8e^x$$

$$\therefore A = -\tfrac{8}{3} \longrightarrow y_p = -\tfrac{8}{3}xe^x \to y = y_c + y_p = c_1e^x + c_2e^{4x} - \tfrac{8}{3}xe^x$$

Ex

Solve $y'' + y = 4x + 10\sin x$

1) $y_c = c_1\cos x + c_2 \sin x$

2) $g(x) = 4x + 10\sin x = g_1(x) + g_2(x)$

$\therefore \quad y_{p_1} = Ax + B$, $y_{p_2} = C\sin x + D\cos x$

But y_c has $\sin x$ term

$\therefore$ in Case II

$\therefore$ y_{p_2} is replaced with $y_{p_2} = Cx\sin x + Dx\cos x$

$\therefore$ We try $y_p = Ax + B + Cx\sin x + Dx\cos x$.

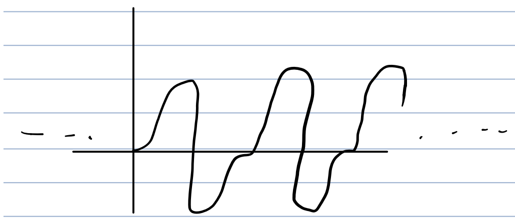
<u>Guessing Solution from Equation</u>

Hint: $y'' + y = 0 \to r^2 + 1 = 0 \to y_c = c_1\cos x + c_2\sin x$.

$\therefore$ y_c is sinusoidal

Graph $\to$ a particular solution $y = y_c + y_p$

$\to$ what $y'' + y = f$ should look like



graph of sum of sinusoids of different periods.

$$y = y_c + y_p = \underbrace{c_1\cos x + c_2\sin x}_{\text{sinusoids of same period}} + \underbrace{y_p}_{\text{needs to be sinusoid of different period.}}$$

y_p behaves like $f \to f$ sinusoid of different period

$\therefore$ $f = \sin\alpha x$ or $\cos\alpha x$, $\alpha \neq 1$.