# 4.3 Homogeneous Linear Equations With Constant Coefficients

Preliminaries:

$\mathbb{R} = (-\infty, +\infty) =$ real numbers

Define formally the symbol $i$ to be the solution to $x^2+1=0$

$\therefore \quad i^2+1=0 \to i^2=-1.$

We formally define the complex numbers $\mathbb{C}$ to be numbers of the form

$$a+bi, \quad a \text{ and } b \text{ in } \mathbb{R}$$

Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

In particular, $e^{\alpha+\beta i} = e^{\alpha}e^{\beta i} = e^{\alpha}(\cos\beta + i\sin\beta)$

Every complex number is of the form $re^{i\theta}$ (radial representation)

The number $a+ib$ and $a-ib$ are called conjugate complex

## Auxiliary Equation

$ay'+by=0$:

Ansatz (educated guess): try $y=e^{mx}$:

$$0 = a(e^{mx})' + be^{mx} = ame^{mx} + be^{mx} = e^{mx}(am+b)$$

$\therefore$ since $e^{mx} \neq 0$, we have $am+b=0 \to m = -\frac{b}{a}$

$\therefore \quad y = e^{-\frac{b}{a}x}$ is a solution.

$ay''+by'+cy=0$

Ansatz: $y=e^{mx}$ worked before, so let's try it again:

$$0 = a(e^{mx})'' + b(e^{mx})' + ce^{mx} = am^2+bm+c$$

$\therefore$ if $m_1, m_2$ are the roots of $am^2+bm+c=0$, then

$$y_1 = e^{m_1x}, \quad y_2 = e^{m_2x}$$

are both solutions.

This idea may be carried to higher order.

## Order Two Case

We consider $ay''+by'+cy=0$ for now...

The equation $am^2+bm+c=0$ is called the auxiliary equation of $ay''+by'+cy=0$.

$m_1, m_2$ roots to $am^2+bm+c=0 \rightarrow m_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}, \; m_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$

Three cases:

- $m_1, m_2$ real and distinct $(b^2-4ac>0)$
- $m_1 = m_2$ real $(b^2-4ac=0)$
- $m_1, m_2$ conjugate complex $(b^2-4ac<0)$ (i.e., $m_1 = a+bi \rightarrow m_2 = a-bi$)

### Case I: Distinct Real Roots

$y_1 = e^{m_1 x}, y_2 = e^{m_2 x}$ both solutions.

Note $W(e^{m_1 x}, e^{m_2 x}) = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix} = m_2 e^{(m_1+m_2)x} - m_1 e^{(m_1+m_2)x}$

$= (m_2 - m_1) e^{(m_1+m_2)x}$

$\neq 0$ since $m_1 \neq m_2$.

$\therefore y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ is the general solution to $ay''+by'+cy=0$.

### Case II: Repeated Real Roots

$m := m_1 = m_2 \rightarrow$ we get one exponential solution $y_1 = e^{mx}$

2nd solution is given by $xe^{mx}$ (check yourself).

$W(e^{mx}, xe^{mx}) = \begin{vmatrix} e^{mx} & xe^{mx} \\ me^{mx} & e^{mx} + mxe^{mx} \end{vmatrix} = e^{2mx} + mxe^{2mx} - mxe^{2mx} = e^{2mx} \neq 0$

$\therefore y = c_1 e^{mx} + c_2 xe^{mx}$ is the general solution

### Case III: Conjugate Complex Roots

$m_1, m_2$ complex conjugate $\Rightarrow$ can write $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$.

$\therefore$ (By case I computations) $y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$ is the general sol.

By Euler's formula:

$$y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$

$$= c_1 e^{\alpha x}(\cos\beta x + i\sin\beta x) + c_2 e^{\alpha x}(\cos\beta x - i\sin\beta x)$$

$$= e^{\alpha x}((c_1+c_2)\cos\beta x + (c_1-c_2)i\sin\beta x)$$

$$:= e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x)$$

where $C_1, C_2$ are arbitrary complex numbers.

Ex Find gen. sol.

a) $2y'' - 5y' - 3y = 0$

aux. eq.: $2m^2 - 5m - 3 = (2m+1)(m-3) = 0 \to m_1 = -\frac{1}{2}, m_2 = 3$

$\therefore$ gen. sol. is $y = c_1 e^{-\frac{1}{2}x} + c_2 e^{3x}$

b) $y'' - 10y' + 25y = 0$

aux eq.: $m^2 - 10m + 25 = (m-5)^2 \to m_1 = m_2 = 5$

$\therefore$ gen. sol. is $y = c_1 e^{5x} + c_2 x e^{5x}$

c) $y'' + 4y' + 7y = 0$

aux. eq.: $m^2 + 4m + 7 = 0$

$$m_1 = \frac{-4+\sqrt{16-28}}{2} = \frac{-4+\sqrt{-12}}{2} = \frac{-4+2i\sqrt{3}}{2} = -2+i\sqrt{3}$$

$m_2 = -2 - i\sqrt{3}$ (conjugate complex)

$\therefore$ $\alpha = -2, \beta = \sqrt{3}$

$\therefore$ gen. sol. is $y = e^{-2x}(c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x)$

## Higher Order Equations

multiplicity of a root: how many times it shows up in factorization:

$(x-1)^2(x-2) \to x=1$ has mult. 2, $x=2$ has mult. 1
$(x+3)^4(x-2-i)^2(x-2+i)^2 \to x=-3$ has mult. 4, $2+i$ has mult. 2, $x=2-i$ has mult. 2.

### nth order Auxiliary Equation

$$(*)\ a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \longrightarrow \underbrace{a_n m^n + a_{n-1}m^{n-1} + \cdots + a_1 m + a_0 = 0}_{\text{Aux. Eq.}}$$

$a_i$: constants

Let $\tilde{r}$ be a root to aux. eq. If $\tilde{r}$ has multiplicity $k$, then $(*)$ has the $k$ lin. indep. sol.

$$e^{\tilde{r}x}, xe^{\tilde{r}x}, \dots, x^{k-1}e^{\tilde{r}x}.$$

The gen. sol. to $(*)$ is the lin. comb. of all such expressions corresponding to each root.

Ex
Solve $y''' + 3y'' - 4y = 0$

Aux: $m^3 + 3m^2 - 4 = (m-1)(m+2)^2$
$m_1 = 1$ has mult. 1 $\to y_1 = e^x$ is a sol
$m_2 = -2$ has mult. 2 $\to y_2 = e^{-2x}, xe^{-2x}$ are sol.

$\therefore$ gen. sol. is

$$y = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}.$$

Ex
Solve $y^{(4)} + 2y'' + y = 0$

Aux eq.: $m^4 + 2m^2 + 1 = (m^2+1)^2 = (m+i)^2(m-i)^2 \to m_1 = -i$ and $m_2 = i$ have mult. 2

$\therefore\ y_1 = e^{-ix}, y_2 = xe^{-ix}, y_3 = e^{ix}, y_4 = xe^{ix}$

are all solution.

$\therefore$ gen. sol. is $y = c_1 e^{-ix} + c_2 x e^{-ix} + c_3 e^{ix} + c_4 x e^{ix}$

$= C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x$

(using Euler's formula)