Chapter 4: Higher Order Differential Equations

4.1 Preliminary Theory - Linear Equations

Initial Value and Boundary Value Problems

nth order IVP:

(*) Solve: $a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$

Subject to: $y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots,\ y^{(n-1)}(x_0) = y_{n-1}$

Theorem (Existence and Uniqueness)

- $a_n, a_{n-1}, \ldots, a_1, a_0, g$ all continuous on an interval I
- $a_n \neq 0$ in I
- x_0 in I

Then (*) has a unique solution.

Ex Find largest interval for the given IVP to have a unique solution according to previous Theorem.

$\begin{cases} xy' + y = 0 \\ y(1) = 1 \end{cases}$

- $a_1(x) = x,\ a_0(x) = 1,\ g(x) = 0$ all continuous
- $a_1(x) = x \neq 0$ on $(-\infty, 0)$ and $(0, +\infty)$
- $x_0 = 1$ belongs to $(0, +\infty)$

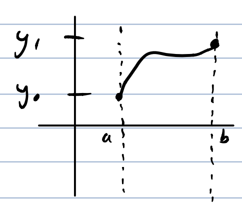
$\therefore$ largest interval is $(0, +\infty)$

Boundary Value Problem:

Solve: $a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$

Subject to: $y(a) = y_0,\ y(b) = y_1$.

Idea:



The boundary values of the solution are specified.

Ex

$x(t) = c_1\cos 4t + c_2 \sin 4t$ solves $x'' + 16x = 0$.

Find solution to the BVPs:

a) $x(0) = 0,\ x(\frac{\pi}{8}) = 0$

$0 = c_1\cos 0 + c_2 \sin 0 = c_1$

$0 = c_1\cos\frac{\pi}{2} + c_2\sin\frac{\pi}{2} = 0\cdot 0 + c_2\cdot 1 = c_2$

$\therefore c_1 = c_2 = 0,\ x(t) = 0.$

This is the only solution from the family

b) $x(0) = 0$, $x(\frac{\pi}{2}) = 1$

$c_1 = 0$ by above

$1 = c_2 \sin 2\pi = c_2 \cdot 0 = 0$

which is a contradiction.

$\therefore$ no solution to this BVP exists.

Some Theory of Linear Equations

Homogeneous Equations

(*) $a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \longleftarrow$ homogeneous

(**) $a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \longleftarrow$ inhomogeneous

$g(x)$ not the zero function

Equation (*) is called the associated homogeneous equation of (**).

This has nothing to do with homogeneous equations from Ch2.

Superposition principle

Theorem (Superposition principle of homogeneous Equations)

- Let $y_1, \ldots, y_k$ be solutions to a hom. linear eq. on I.

Then the linear combination

$$y = c_1y_1 + c_2y_2 + \cdots + c_ky_k, \quad c_i \text{ arbitrary constant.}$$

is also a solution to $Ly = 0$ on I.

Proof (Idea):

We consider $n = k = 2$:

Suppose y_1, y_2 solve

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Then $a_2(x)\frac{d^2y}{dx^2} = a_2(x)\frac{d^2}{dx^2}(c_1y_1 + c_2y_2) = c_1a_2(x)\frac{d^2y_1}{dx^2} + c_2a_2(x)\frac{d^2y_2}{dx^2}$

$a_1(x)\frac{dy}{dx} = a_1(x)\frac{d}{dx}(c_1y_1 + c_2y_2) = c_1a_1(x)\frac{dy_1}{dx} + c_2a_1(x)\frac{dy_2}{dx}$

$a_0(x)y = a_0(x)(c_1y_1 + c_2y_2) = c_1a_0(x)y_1 + c_2a_0(x)y_2$

$$\therefore \quad a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = c_1 a_2(x)\frac{d^2y_1}{dx^2} + c_1 a_1(x)\frac{dy_1}{dx} + c_1 a_0(x)\frac{dy_1}{dx}$$

$$+ c_2 a_2(x)\frac{d^2y_2}{dx^2} + c_2 a_1(x)\frac{dy_2}{dx} + c_2 a_0(x)\frac{dy_2}{dx}$$

$$= 0$$

since y_1, y_2 are solution.

For other k, n, the proof is similar.

Ex

e^x and e^{-x} solve $y'' - y = 0$ on $(-\infty, +\infty)$: $(e^{\pm x})'' - e^{\pm x} = e^{\pm x} - e^{\pm x} = 0$

$\therefore$ by superposition principle, $c_1e^x + c_2e^{-x}$ solves $y'' - y = 0$:

$$(c_1e^x + c_2e^{-x})'' - (c_1e^x + c_2e^{-x}) = c_1e^x + c_2e^{-x} - c_1e^x - c_2e^{-x} = 0.$$

Linear Dependence and Linear Independence.

Ex

a) $y_1 = x + 1$, $y_2 = x^2$, $y_3 = 3x^2 + 2x + 2$.

Q: can we write y_3 as a linear combination of y_1, y_2, i.e.,

$$y_3 = c_1y_1 + c_2y_2\ ?$$

We want $y_3 = c_1x + c_1 + c_2x^2 = 3x^2 + 2x + 2 \rightarrow c_2 = 3,\ c_1 = 2$

$\therefore$

$$y_3 = 2y_1 + 3y_2.$$

Thus there is a "linear dependence" among these functions, namely, y_3 may be written as a linear combination of y_1, y_2.

Put another way, there are constants c_1, c_2, c_3, not all zero, such that

$$c_1y_1 + c_2y_2 + c_3y_3 = 0 \quad (c_1 = 2,\ c_2 = 3,\ c_3 = -1).$$

Ex

Can the same be done for $y_1 = \cos^2 x$, $y_2 = \sin^2 x$, $y_3 = x$?

$c_1\cos^2 x + c_2\sin^2 x + c_3 x = 0 \rightarrow c_1\cos^2 0 + c_2\sin^2 0 + c_3 0 = c_1 = 0$

$\rightarrow c_2 \sin \pi + c_3\pi = c_3\pi = 0 \rightarrow c_3 = 0$

$\rightarrow c_2\sin^2\frac{\pi}{2} = c_2 = 0.$

$\therefore$ No!

How about $y_1 = \cos^2 x$, $y_2 = \sin^2 x$, $y_3 = x$, $y_4 = 1$?

Let $c_1 = c_2 = 1$, $c_3 = 0$, $c_4 = -1$:

$$c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4 = \cos^2 x + \sin^2 x - 1 = 0$$

which is true.

A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be linearly dependent on an interval I. if there exists constants $c_1, \ldots, c_n$, not all 0, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for every x in the interval. If the set is not linearly dependent, they are called linearly independent.

Ex f_1, f_2 linearly dependent on $I \to$

$$c_1 f_1(x) + c_2 f_2(x) = 0, \quad x \text{ in } I$$
c_1, c_2 some constants.

$\therefore \quad f_1(x) = -\frac{c_2}{c_1} f_2(x)$

i.e., if f_1, f_2 are lin. dep, then they are multiples of each other.

Ex

a) $y_1 = x$, $y_2 = |x|$ are not lin. dep.:

$c_1 x + c_2 |x| = 0 \to \quad c_1(-1) + c_2|-1| = -c_1 + c_2 = 0$
$\qquad c_1 \cdot 1 + c_2 \cdot |1| = c_1 + c_2 = 0$
$\qquad \therefore \quad c_1 = c_2 = 0.$

b) $y_1 = x$, $y_2 = 2x$, $y_3 = |x|$ are lin. dep.

$$-2y_1 + y_2 + 0 \cdot y_3 = -2x + 2x + 0 = 0$$
So $c_1 = -2$, $c_2 = 1$, $c_3 = 0$.

Wronskians

$W(f_1) = |f_1|$

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = f_1 \cdot f_2' - f_1' f_2$$

$$\begin{array}{cc} - & + \end{array}$$

$$W(f_1, f_2, f_3) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = f_1 f_2' f_3'' + f_2 f_3' f_1'' + f_3 f_1' f_2'' - f_1'' f_2' f_3 - f_2'' f_3' f_1 - f_3'' f_1' f_2$$

$$\begin{array}{ccccc} f_1 & f_2 & f_3 & f_1 & f_2 \\ f_1' & f_2' & f_3' & f_1' & f_2' \\ f_1'' & f_2'' & f_3'' & f_1'' & f_2'' \\ - & - & - & + & + & + \end{array}$$

$W(f_1, f_2, \dots, f_k) = \dots$

Ex

Find the Wronskian of the following set of functions

a) $f_1(x) = e^x$, $f_2(x) = e^{-x}$

a) $$W(e^x, e^{-x}) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = e^x \cdot (-e^{-x}) - e^x \cdot e^{-x} = -2$$

$$\begin{array}{cc} - & + \end{array}$$

b) $f_1(x) = e^x$, $f_2(x) = e^{-x}$, $f_3(x) = x$

b) $$W(e^x, e^{-x}, x) = \begin{vmatrix} e^x & e^{-x} & x \\ e^x & -e^{-x} & 1 \\ e^x & e^{-x} & 0 \end{vmatrix} = e^x(-e^{-x}) \cdot 0 + e^{-x} \cdot 1 \cdot e^x + x e^x e^{-x} - e^x(-e^{-x})x - e^{-x} \cdot 1 \cdot e^x - 0 \cdot e^x \cdot e^{-x}$$

$$= 0 + 1 + x + x - 1 = 2x$$

$$\begin{array}{ccccc} e^x & e^{-x} & x & e^x & e^{-x} \\ e^x & -e^{-x} & 1 & e^x & -e^{-x} \\ e^x & e^{-x} & 0 & e^x & e^{-x} \\ - & - & - & + & + & + \end{array}$$

Theorem (Criterion for linearly independent solution)

- $y_1, \dots, y_n$ n sol. to nth order hom. lin. ODE on I

Then $y_1, \dots, y_n$ are lin. indep on I iff $W(y_1, \dots, y_n) \neq 0$ for at least one x in I.
Actually, $W(y_1, \dots, y_n) \neq 0$ for a single x in I is enough.

Ex

a) e^{2x}, e^{-x} solve $y'' - y' - 2y = 0$. Are they lin. indep on $(-\infty, \infty)$?

$y_1 = e^{2x}, y_2 = e^{-x}$

$$W(e^{2x}, e^{-x}) = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-x} \\ 2e^{2x} & -e^{-x} \end{vmatrix}$$

$$= -e^{x} - 2e^{x}$$

$$= -3e^{x} \neq 0$$

$\therefore$ lin. indep.

b) e^{-x}, e^{x}, e^{-2x} solve $y''' + 2y'' - y' - 2y = 0$

$$W(e^{-x}, e^{x}, e^{-2x}) = \begin{vmatrix} e^{-x} & e^{x} & e^{-2x} \\ -e^{-x} & e^{x} & -2e^{-2x} \\ e^{-x} & e^{x} & 4e^{-2x} \end{vmatrix} = e^{-x} e^{x} 4e^{-2x} + e^{x}(-2e^{-2x})e^{-x} + e^{-2x}(-e^{-x})e^{x} - e^{-x}e^{x}e^{-2x} - e^{x}(-2e^{-2x})e^{-x} - 4e^{-2x}(-e^{-x})e^{x}$$

$$\begin{array}{ccccc} e^{-x} & e^{x} & e^{-2x} & e^{-x} & e^{x} \\ -e^{-x} & e^{x} & -2e^{-2x} & -e^{-x} & e^{x} \\ e^{-x} & e^{x} & 4e^{-2x} & e^{-x} & e^{x} \\ - & - & - & + \quad + & + \end{array}$$

$$= 4e^{-2x} - 2e^{-2x} - e^{-2x} - e^{-2x} + 2e^{-2x} + 4e^{-2x}$$

$$= 6e^{-2x}$$

Fundamental Set of Solutions:

Any set $y_1, \dots, y_n$ of n lin. indep. sol. of hom. lin. nth order ODE on an interval I is said to be a fundamental set of solution (FSoS) on I.

Theorem a FSoS always exists on an interval.

Theorem (General Solution)

- $y_1, \dots, y_n$ a FSoS to an nth order hom. lin. ODE on I.

Then $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ is the general solution on I.

Proof: Follows from uniqueness theorem.

Ex

a) e^{3x}, e^{-3x} solve $y''-9y=0$ on $(-\infty,\infty)$. Find general solution on $(-\infty,\infty)$

$n=2$, we check if $y_1=e^{3x}, y_2=e^{-3x}$ are lin. indep.:

$$W(e^{3x},e^{-3x}) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -3-3=-6\neq 0 \text{ on } (-\infty,\infty).$$

$\therefore$ y_1, y_2 is a F.S.S to $y''-9y=0$.

$\therefore$ by Theorem, $y=c_1y_1+c_2y_2 = c_1e^{3x}+c_2e^{-3x}$ is the general solution on I.

b) e^{x}, e^{2x}, e^{3x} solve $y'''-6y''+11y'-6y=0$. Find the gen. solution on $(-\infty,+\infty)$.

$$W(e^{x},e^{2x},e^{3x}) = \begin{vmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x} \end{vmatrix} = 18e^{6x}+3e^{6x}+4e^{6x}-2e^{6x}-12e^{6x}-9e^{6x}$$

$$= 2e^{6x} \neq 0 \text{ on } (-\infty,+\infty)$$

$$\begin{array}{ccccc} e^{x} & e^{2x} & e^{3x} & e^{x} & e^{2x} \\ e^{x} & 2e^{2x} & 3e^{3x} & e^{x} & 2e^{2x} \\ e^{x} & 4e^{2x} & 9e^{3x} & e^{x} & 4e^{2x} \end{array}$$

$- \quad - \quad - \quad + \quad + \quad +$

$\therefore$ e^{x}, e^{2x}, e^{3x} is a F.S.S, and $y=c_1e^{x}+c_2e^{2x}+c_3e^{3x}$ is the gen. sol. on $(-\infty,+\infty)$.

4.3.1 Nonhomogeneous Equations (aka Inhomogeneous Equations)

We will come back to this idea in 4.4 and 4.6.