Math 2410 # Practice Exam II **March x^{th}, 2019**

Name: ________________________________

Instructions:

- Answer each question to the best of your ability. Show your work or receive no credit.
- All answers must be written clearly. Be sure to erase or cross out any work that you do not want graded. Partial credit can not be awarded unless there is legible work to assess.
- If you require extra space for any answer, you may use the back sides of the exam pages. Please indicate when you have done this so that I do not miss any of your work.

Academic Integrity Agreement

I certify that all work given in this examination is my own and that, to my knowledge, has not been used by anyone besides myself to their personal advantage. Further, I assert that this examination was taken in accordance with the academic integrity policies of the University of Connecticut.

Signed:________________________________
(full name)

Questions:	1	2	3	4	5	**Total**
Score:						

1. (5 points) Use the method of undetermined coefficients to find the general solution of

$$y'' - 16y = 2e^{4x} \tag{1}$$

aux. eq. $r^2 - 16 = 0 \rightarrow r = \pm 4$

$\therefore \quad y_c = c_1 e^{4x} + c_2 e^{-4x}$

$f(x) = 2e^{4x} \rightarrow y_p = Ae^{4x}$, but y_c has a e^{4x} term

$\therefore$ we replace y_p with $y_p = Axe^{4x}$.

$\therefore \quad y_p' = Ae^{4x} + 4Axe^{4x}$

$y_p'' = 8Ae^{4x} + 16Axe^{4x}$

$\therefore \quad y_p'' - 16y_p = 2e^{4x} \rightarrow 8Ae^{4x} + 16Axe^{4x} - 16Axe^{4x} = 2e^{4x}$

$\rightarrow \quad 8Ae^{4x} = 2e^{4x}$

$\therefore \quad A = \frac{1}{4}$

$\therefore \quad y_p = \frac{1}{4} xe^{4x}$

$\therefore$ gen. sol. is $\quad y = y_c + y_p$

2. (5 points) A mass of 1 slug, when attached to a spring, stretches it 32/9 feet and then it comes to rest in the equilibrium position. Starting at $t = 0$, an external force equal to $f(t) = \sin(3t)$ is applied to the system. There is no damping on the system.

   (a) Set up the IVP governing this spring-mass system.

   (b) Using variation of parameters, find the corresponding equation of motion.

   (c) Explain why or why not the mass will ever come back to rest.

(a) mass $m = 1$

weight $W = mg = 1 \times 32 = 32$

spring constant $K = \frac{W}{\text{length stretched}} = \frac{32}{32/9} = 9$

$x(0) = 0$

$x'(0) = 0$

$f(t) = \sin(3t)$

$\beta = 0$ (no damping)

$mx'' + \beta x' + Kx = f(t)$

$\downarrow$

$\therefore \begin{cases} x'' + 9x = \sin(3t) \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$

(b) aux. eq.: $m^2 + 9 = 0 \to m = \pm 3i \to x_c(t) = c_1 \cos(3t) + c_2 \sin(3t) = c_1 x_1 + c_2 x_2$

particular solution: $x_p = u_1 x_1 + u_2 x_2$

$W(x_1, x_2) = \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix} = 3$

$u_1' = \frac{-x_2 f}{W(x_1, x_2)} = -\frac{\sin^2(3t)}{3} \to \quad u_1 = -\frac{1}{6}t + \frac{1}{36}\sin(6t)$

$u_2' = \frac{x_1 f}{W(x_1, x_2)} = \frac{\cos(3t)\sin(3t)}{3} \quad u_2 = -\frac{1}{36}\cos(6t)$

$\therefore x_p = -\frac{1}{6}t\cos(3t) + \frac{1}{36}\sin(6t)\cos(3t) - \frac{1}{36}\cos(6t)\sin(3t)$

$= -\frac{1}{6}t\cos(3t) + \frac{1}{36}\left(2\sin(3t)\cos^2(3t) - 2\sin(3t)\cos^2(3t) + \sin(3t)\right)$

$= -\frac{1}{6}t\cos(3t) + \frac{1}{36}\sin(3t)$

$\therefore x = c_1\cos(3t) + c_2\sin(3t) - \frac{1}{6}t\cos(3t) + \frac{1}{36}\sin(3t)$

$= c_1\cos(3t) + c_2\sin(3t) - \frac{1}{6}t\cos(3t)$ (c_2 relabeled)

$x(0) = 0 \to c_1 = 0$

$x'(t) = 3c_2\cos(3t) - \frac{1}{6}\cos(3t) + \frac{1}{2}t\sin(3t)$

$x'(0) = 0 \to 0 = 3c_2 - \frac{1}{6} \to c_2 = \frac{1}{18}$

$\therefore x(t) = \frac{1}{18}\sin(3t) - \frac{1}{6}t\cos(3t)$

c) long term: system oscillates uncontrolably

3. (5 points) Consider the following linear homogeneous differential equation:

$$a(x)y'' + b(x)y' + c(x)y = 0.$$

(a) If y_1, y_2 form a fundamental set of solutions, then what can we say about the value of their Wronskian $W(y_1, y_2)$?

(b) If f and g are solutions, show by direct computation that $f + g$ is also a solution.

(c) True or False: if Y_1, Y_2, Y_3 are three solutions, then we may write one of them as a linear combination of the remaining two. Justify by using the definition of linear dependence and the theory of linear differential equations.

(a) y_1, y_2 a fsos $\Rightarrow$ y_1, y_2 lin. indep.

$\therefore$ $W(y_1, y_2) \neq 0$

(b) $af'' + bf' + cf = 0$

$ag'' + bg' + cg = 0$

$$\therefore \quad a(f+g)'' + b(f+g)' + c(f+g) = af'' + bf' + cf + ag'' + bg' + cg$$
$$= 0 + 0$$
$$= 0$$

$\therefore$ $f+g$ is a solution.

(c) True: order of ODE is 2.

$\therefore$ Y_1, Y_2, Y_3 have to be lin. dep.

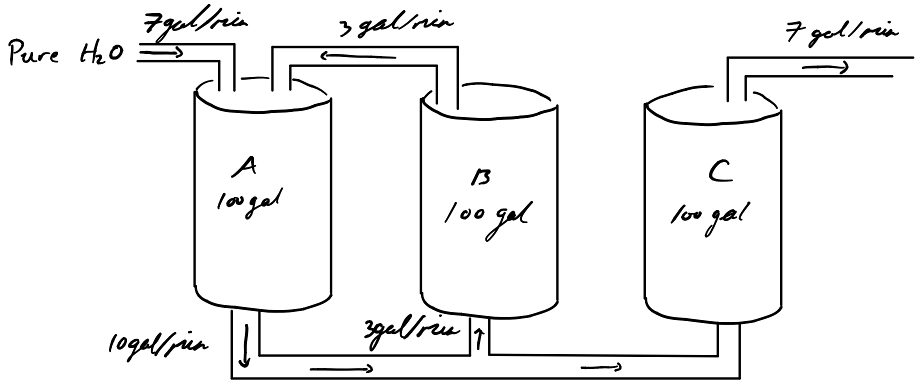
$\therefore$ $c_1Y_1 + c_2Y_2 + c_3Y_3 = 0$ for some c_1, c_2, c_3 not all zero.

$\therefore$ Without loss of generality, assume $c_1 \neq 0$. Then

$$Y_1 = \frac{c_2}{c_1} Y_2 + \frac{c_3}{c_1} Y_3 .$$

4. (5 points) Consider the following 100 gallon tanks containing brine. As done in class, construct a system of differential equations which model the salt contents $x_A(t), x_B(t)$, and $x_C(t)$ of tanks A, B, and C, respectively, given the indicated information.

What values should $x_A(t)$, $x_B(t)$, and $x_C(t)$ respectively approach as $t \to \infty$?



$$x_A' = 0 \times 7 + \frac{x_B}{100} \times 3 - \frac{x_A}{100} \times 10$$

$$= \frac{3}{100} x_B - \frac{1}{10} x_A$$

$$x_B' = \frac{x_A}{100} \times 3 - \frac{x_B}{100} \times 3$$

$$= \frac{3}{100} x_A - \frac{3}{100} x_B$$

$$x_C' = \frac{x_A}{100} \times 7 - \frac{x_C}{100} \times 7$$

$$= \frac{7}{100} x_A - \frac{7}{100} x_C$$

5. (5 points) Solve the following nonlinear ODE

$$y'''(y''+y)(y''-y)=0. \tag{2}$$

Note: there will be three different solutions.

$y'''(y''+y)(y''-y)=0 \longrightarrow y'''=0$ or $y''+y=0$ or $y''-y=0$

$y'''=0 \rightarrow m^3=0 \rightarrow m_1=m_2=m_3=0 \rightarrow y_1 = c_1 + c_2 x + c_3 x^2$

$y''+y=0 \rightarrow m^2+1=0 \rightarrow m=\pm i \rightarrow y_2 = c_1 e^{ix} + c_2 e^{-ix}$

or $= c_1 \cos x + c_2 \sin x$

$y''-y=0 \rightarrow m^2-1=0 \rightarrow m=\pm 1 \rightarrow y_3 = c_1 e^{x} + c_2 e^{-x}.$