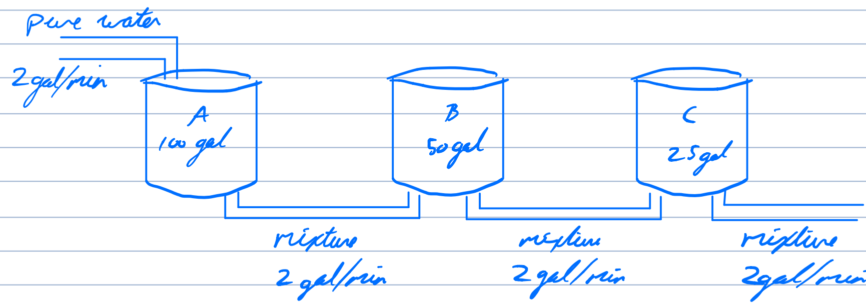
3.3 Modeling With Systems of 1st Order ODEs

Idea: See how systems of ODEs naturally arise in modeling

What is a system of ODEs?

Mixtures

Ex
Three tanks contain brine:



$x(t)$ = salt lbs in A
$y(t)$ = salt lbs in B
$z(t)$ = salt lbs in C

Goal: Find ODEs that model amount of salt in each tank.

A: $\frac{dx}{dt} = R_{in} - R_{out}$

$R_{in} = 0$ since pure water is flowing in.

$R_{out} = \begin{pmatrix}\text{Density of salt}\\ \text{leaving}\end{pmatrix} \times \begin{pmatrix}\text{Rate of Volume}\\ \text{leaving}\end{pmatrix}$

$= \frac{x(t)}{100} \times 2$

$= \frac{x(t)}{50}$

$\therefore \frac{dx}{dt} = \frac{x(t)}{50}$

B: $\frac{dy}{dt} = R_{in} - R_{out}$

R_{in} = Rate of salt leaving A $= \frac{x(t)}{50}$

$R_{out} = \begin{pmatrix}\text{Density of salt}\\ \text{leaving}\end{pmatrix} \times \begin{pmatrix}\text{Rate of Volume}\\ \text{leaving}\end{pmatrix}$

$= \frac{y(t)}{50} \times 2$

$= \frac{2y(t)}{50}$

$\therefore \frac{dy}{dt} = \frac{x(t)}{50} - \frac{2y(t)}{50}$

C: $\frac{dz}{dt} = R_{in} - R_{out}$

R_{in} = Rate of salt leaving B $= \frac{2y(t)}{50}$

$R_{out} = \begin{pmatrix}\text{Density of salt}\\ \text{leaving}\end{pmatrix} \times \begin{pmatrix}\text{Rate of volume}\\ \text{leaving}\end{pmatrix}$

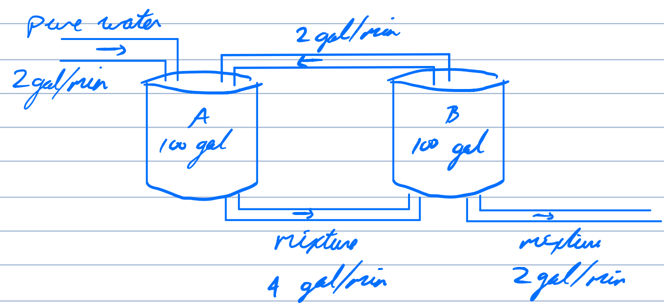
$= \frac{z(t)}{25} \times 2$

$= \frac{2z(t)}{25}$

$\therefore \quad \frac{dy}{dt} = \frac{2y(t)}{50} - \frac{2z(t)}{25}$

Ex

Two tanks contain brine



$x_1(t)$ = amount of salt in A
$x_2(t)$ = " " B

Construct a system of ODEs modeling how x_1 and x_2 change over time.

Solution

A: $\frac{dx_1}{dt} = R_{in} - R_{out}$

R_{in}: two sources of input:

1) pure water of density 0 at rate 2gal/min
2) brine from B of density $\frac{x_2(t)}{100}$ at rate 2gal/min

$\therefore \quad R_{in} = 0 \cdot 2 + \frac{x_2(t)}{100} \cdot 2 = \frac{x_2(t)}{50}$

R_{out}: one output:

1) brine from A of density $\frac{x_1(t)}{100}$ at rate 2gal/min.

$\therefore \quad R_{out} = \frac{x_1(t)}{100} \cdot 4 = \frac{x_1(t)}{25}$

$\therefore \quad \frac{dx_1}{dt} = R_{in} - R_{out} = \frac{x_2(t)}{50} - \frac{x_1(t)}{25}$

B: $\frac{dx_2}{dt} = R_{in} - R_{out}$

R_{in}: One input

1) brine from A of density $\frac{x_1(t)}{100}$ at rate 2 gal/min

$\therefore \quad R_{in} = \frac{x_1(t)}{100} \cdot 4 = \frac{x_1(t)}{25}$

R_{out}: two outputs

1) $B \to A$ of density $\frac{x_2(t)}{100}$ at rate 2 gal/min

2) $B \to$ out of density $\frac{x_2(t)}{100}$ at rate 2 gal/min

$\therefore \quad R_{out} = \frac{x_2(t)}{100} \cdot 2 + \frac{x_2(t)}{100} \cdot 2 = \frac{x_2(t)}{25}.$

$\therefore \quad \frac{dx_2}{dt} = R_{in} - R_{out} = \frac{x_1(t)}{50} - \frac{x_2(t)}{25}$

$\therefore$ System is

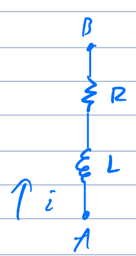
$$\frac{dx_1}{dt} = \frac{x_2(t)}{50} - \frac{x_1(t)}{25}$$
$$\frac{dx_2}{dt} = \frac{x_1(t)}{25} - \frac{x_2(t)}{25}.$$

LR Circuits

Resistor: R = resistance, $\uparrow i$ = current $\longrightarrow$ voltage drop across resistor $= i \cdot R$

Inductor: L = inductance, $\uparrow i$ = current $\longrightarrow$ voltage drop across inductor $= L\frac{di}{dt}$

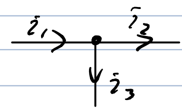
Ex



voltage drop from A to B:

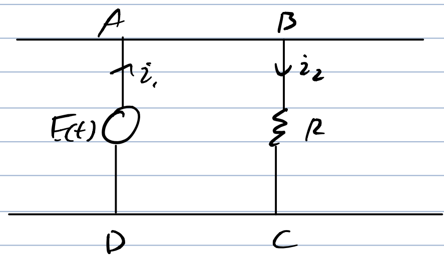
$L\frac{di}{dt} + Ri.$

Current through a junction :



$\Longrightarrow$ $i_1 = i_2 + i_3$.

Voltage in a loop sums to 0 :

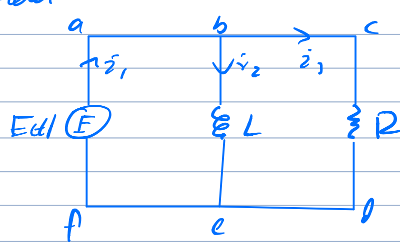


Voltage in loop ABCD : E(t) adds E(t) voltage, whereas, R drops voltage by $i_2 R$.

$\therefore \quad E(t) - i_2 R = 0 \quad \to \quad E(t) = i_2 R$.

Ex

Consider



loop abcdef :

$0 =$ sum of voltage $= E(t) - i_3 R$

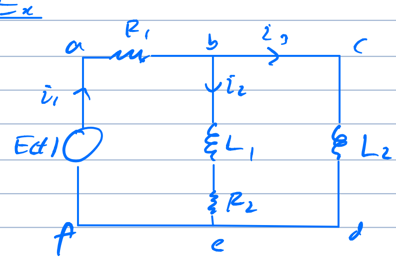
loop abef :

$0 =$ sum of voltage $= E(t) - L\frac{di_2}{dt}$

conservation of current:

$i_1 = i_2 + i_3$

Ex



Use a system in i_2, i_3 to model the current passing through the circuit.

current: $i_1 = i_2 + i_3$

loop abcdef : $E(t) - R_1 i_1 - \frac{di_3}{dt} L_2$

loop abef : $E(t) - R_1 i_1 - \frac{di_2}{dt} L_1 - i_2 R_2$

$\therefore \quad E(t) = R_1 i_1 - \frac{di_3}{dt} L_2 = R_1(i_2 + i_3) + \frac{di_3}{dt} L_2$

$E(t) = R_1(i_2 + i_3) + \frac{di_2}{dt} L_1 + i_2 R_2$