3 Modeling with First Order Differential Equations

3.1 Linear Models

Idea: Use linear ODE to model application which are linear in nature.

Examples are taken from handout.

Example 1:

Half-Life: time it takes for one-half of atoms in a sample of some element to decay into atoms of another element.

$A(t)$ = amount at time t
A_0 = initial amount = $A(0)$

ODE: $\frac{dA}{dt} = kA(t)$
Solution: linear $\Rightarrow \frac{dA}{dt} - kA(t) = 0 \rightarrow \mu = e^{\int -k\,dt} = e^{-kt} \rightarrow A = \frac{1}{\mu}\int \mu f\,dt + \frac{c}{\mu} = ce^{kt}$.

$A_0 = A(0) = ce^{k\cdot 0} = c \rightarrow A(t) = A_0 e^{kt}$.

$A(15) = A_0 - 0.00043A_0 = 0.99957A_0$

$\therefore\ 0.99957A_0 = A_0 e^{k\cdot 15} \rightarrow k = -2.867\times 10^{-5}$

Half-life is t s.t. $A(t) = \frac{1}{2}A_0$

$\therefore\ \frac{1}{2}A_0 = A_0 e^{-2.867\times 10^{-5}t} \rightarrow t \cong 24{,}180$ years = half-life.

Example 2:

$T(t)$ = temp.
$T_0 = T(0)$ = initial temp = 300
T_m = ambient temp = 70
$T(3) = 200$

ODE: $\frac{dT}{dt} = k(T - T_m) = k(T-70)$
Solution: Separable $\rightarrow \frac{dT}{k(T-70)} = dt \rightarrow \frac{1}{k}\ln(T-70) = t \rightarrow T = 70 + ce^{kt}$

$300 = T(0) = 70 + c \rightarrow c = 230 \rightarrow T = 70 + 230e^{kt}$

$200 = T(3) = 70 + 230e^{k\cdot 3} \rightarrow k = -0.19018$.

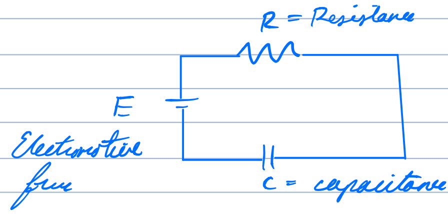
$\therefore\ T = 70 + 230e^{-0.19018t}$

Want t s.t. $T(t) = 80$.

$\therefore\ 80 = T(t) = 70 + 230e^{-0.19018t} \rightarrow t \cong 16.487$

Example 3

RC-Series Circuit:



ODE: $R\frac{dq}{dt} + \frac{1}{C}q = E(t)$

$i = \frac{dq}{dt}$

$E = 100$

$R = 100 \qquad \therefore \quad 100\frac{dq}{dt} + q = 100$

$C = 1$

$i(0) = 1$

Solution: Linear $\to \frac{dq}{dt} + \frac{1}{100}q = 1 \to \mu = e^{\int \frac{1}{100}dt} = e^{\frac{t}{100}} \to q = \frac{1}{\mu}\int \mu f dt + \frac{c}{\mu} = e^{-\frac{t}{100}}\int e^{\frac{t}{100}}dt + ce^{-\frac{t}{100}}$

$= 100 + ce^{-t/100}.$

$i = \frac{dq}{dt} = -\frac{1}{100}ce^{-t/100} \to 1 = i(0) = -\frac{1}{100}c \to c = -100$

$\therefore \quad q(t) = 100 - 100e^{-t/100}$

$\lim_{t\to\infty} q(t) = 100.$