2.6 Euler's Method

Idea: Use tangent line approximation/linearization to approximately solve an IVP.

We approximate a solution to

$$\begin{cases} y'(x) = f(x,y) \\ y(x_0) = y_0 \end{cases}, \quad f \text{ is continuous.}$$

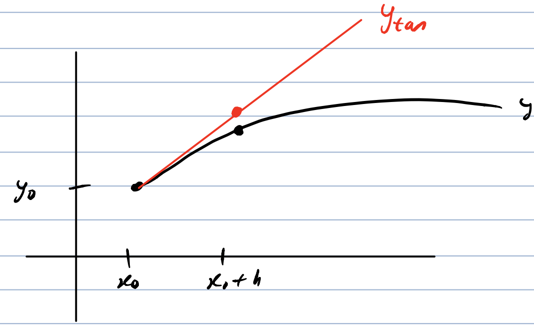
Key Observation 1)

$$y'(x_0) = f(x_0,y_0) = \text{slope of solution passing } (x_0,y_0):$$

Key Observation 2)

$$y_{tan} = y'(x_0)(x-x_0)+y_0 = f(x_0,y_0)(x-x_0)+y_0$$

approximate y near (x_0,y_0)

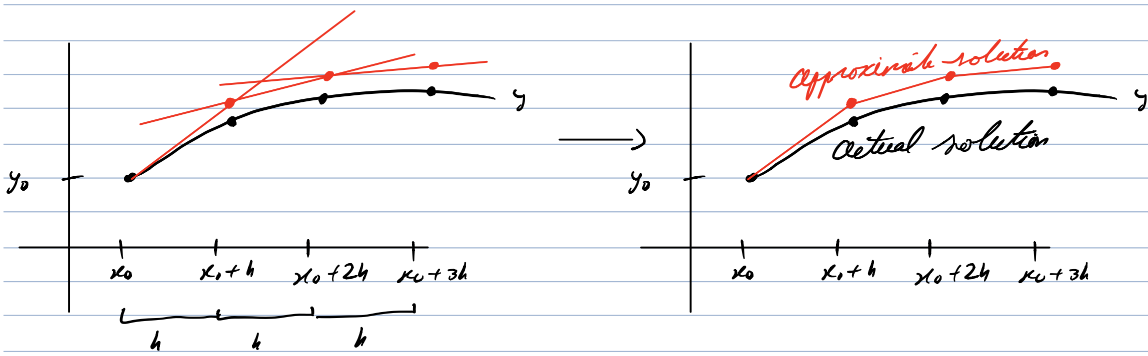


$|y_{tan}(x_0+h) - y(x_0+h)|$ is small for small h.

$\therefore$ Knowing $f(x,y) \Rightarrow$ can approximate y via tangent line.

Key Observation 3)

Continuity of $f \Rightarrow$ Can iterate the approximation



Then we obtain an approximate solution which is piecewise linear.

Remarks

1) This approximation picks up many errors

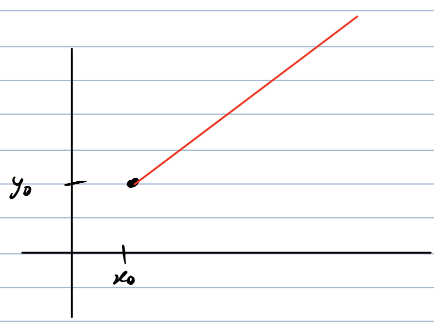
2) One can smooth out approximate solution

3) small step size $h \longrightarrow$ more computation + accuracy.

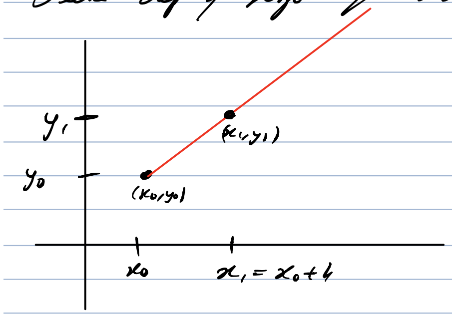
Pseudo code:

Fix step size h, x_0, y_0.

Plot $y_{tan} = f(x_0, y_0)(x - x_0) + y_0$



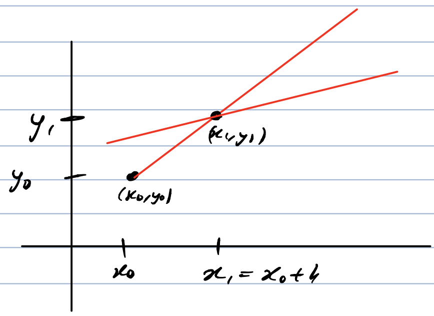
Take step of size h from x_0 to $x_1 = x_0 + h$:



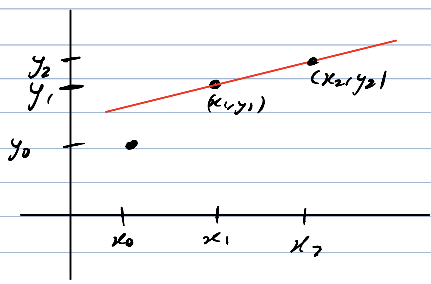
$\longrightarrow$ $y_1 = y_{tan}(x_0+h) = f(x_0,y_0)(x_0+h-x_0) + y_0$
$= f(x_0,y_0)h + y_0$

Note $y_1 \sim y(x_1)$

Plot y_{tan} passing through (x_1, y_1)



Take step of size h from $x_1 = x_0 + h$ to $x_2 = x_1 + h = x_0 + 2h$



$y_2 = y_{tan}(x_0+2h) = f(x_1,y_1)(x_1+h-x_1) + y_1$
$= f(x_1,y_1)h + y_1$

Note $y_2 \sim y(x_2)$

Continue as desired:

Euler's Method in summary:

$y_j = f(x_{j-1}, y_{j-1})h + y_{j-1}$ $\longleftarrow$ compute y_j first.
$x_j = x_{j-1} + h = x_0 + jh$

approximate a solution to $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$

with $y(x_j)$ approximated by y_j

Ex Use Euler's Method to approximate a solution to the IVP

$$y' = xy$$
$$y(1) = 1$$

with $h = 0.5$ for 4 steps.

We use Euler's method:

$$\begin{cases} x_0 = 1, \ y_0 = 1 \\ x_j = x_{j-1} + h \\ y_j = f(x_{j-1}, y_{j-1})h + y_{j-1} = x_{j-1} \cdot y_{j-1} h + y_{j-1} \end{cases}$$

$y_0 = 1 \to y_1 = x_0 y_0 h + y_0 = 1 \cdot 1 \cdot \frac{1}{2} + 1 = 1.5$
$x_0 = 1 \to x_1 = x_0 + h = 1 + \frac{1}{2} = 1.5$

$y_1 = 1.5 \to y_2 = x_1 y_1 h + y_1 = 1.5 \cdot 1.5 \cdot \frac{1}{2} + 1.5 = 2.625$
$x_1 = 1.5 \to x_2 = x_1 + h = 1.5 + 0.5 = 2$

$y_2 = 2.625 \to y_3 = x_2 y_2 + y_2 = 2 \cdot 2.625 \cdot \frac{1}{2} + 2.625 = \cancel{7.875}\ 5.25$
$x_2 = 2 \to x_3 = x_2 + h = 2 + 0.5 = 2.5$

$y_3 = 7.875 \to y_4 = x_3 y_3 + y_3 = 2.5 \cdot 7.875 \cdot \frac{1}{2} + 7.875 = \cancel{27.5625}\ 11.8125$
$x_3 = 2.5 \to x_4 = x_3 + h = 2.5 + 0.5 = 3$

So
$y(1) = 1$
$y(1.5) \sim y_1 = 1.5$
$y(2) \sim y_2 = 2.625$
$y(2.5) \sim y_3 = \cancel{7.875}\ 5.25$
$y(3) \sim y_4 = \cancel{27.5625}\ 11.8125$

Solving $y' = xy \to \frac{dy}{y} = x\,dx \to \ln y = \frac{1}{2}x^2 + c \to y = ce^{\frac{1}{2}x^2}$
$y(1) = 1 \to 1 = ce^{\frac{1}{2}} \to c = e^{-\frac{1}{2}}$
$y = e^{\frac{1}{2}(x^2 - 1)}$.

Actual values
$y(1) = 1$
$y(1.5) \sim 1.868$
$y(2) \sim 4.481$
$y(2.5) \sim 13.804$
$y(3) \sim 54.598$

So the approximation is not so great far away from $(x_0, y_0) = (1, 1)$. This is because the step size $h = \frac{1}{2}$ is large. If $h = 0.0005$, the approximation would be really good!

Ex Use $h=1$ in Euler's method to estimate $y(3)$ if y solves

$$\begin{cases} y' = x^2+y^2 \\ y(0) = 1 \end{cases}$$

Euler's Method: $y_j = f(x_{j-1}, y_{j-1})h + y_{j-1}$

$x_0 = 0$

$y_0 = 1$ Want $y(3)$

$h = 1$

$f(x,y) = x^2+y^2$

$y_1 = f(x_0, y_0)h + y_0 = f(0,1)\cdot 1 + 1 = 0^2+1^2+1 = 2$

$x_1 = x_0 + h = 0+1 = 1$

$y(x_1) = y(1) \sim y_1 = 2$

$y_2 = f(x_1, y_1)\cdot h + y_1 = f(1,2)\cdot 1 + 2 = 1^2+2^2+2 = 7$

$x_2 = x_1 + h = 1+1 = 2$

$y(x_2) = y(2) \sim y_2 = 7$

$y_3 = f(x_2, y_2)\cdot h + y_2 = f(2,7)\cdot 1 + 7 = 2^2+7^2+7 = 60$

$x_3 = x_2 + h = 3$

$y(x_3) = y(3) \sim y_3 = 60$.