2.4 Exact Equations

(!) Partial Differentiation: $\frac{\partial}{\partial x} f = f_x$ - diff. wrt x $\frac{\partial}{\partial x}(x^2y + e^y) = 2xy$

$\frac{\partial}{\partial y} f = f_y$ - diff. wrt y $\frac{\partial}{\partial y}(x^2y + e^y) = x^2 + e^y$

(!) Partial Integration:

$$\frac{\partial}{\partial x} G(x,y) = f(x,y) \Longrightarrow \int f(x,y)\,dx = G(x,y) + g(y)$$
$$\frac{\partial}{\partial y} H(x,y) = f(x,y) \Longrightarrow \int f(x,y)\,dy = H(x,y) + h(x)$$

"constants of integration."

Why? Observe $\frac{\partial}{\partial x}(G(x,y) + h(y)) = \frac{\partial}{\partial x} G(x,y) + 0 \Longrightarrow G(x,y) + h(y)$ an x-anti-derivative

$\frac{\partial}{\partial y}(H(x,y) + g(x)) = \frac{\partial}{\partial y} H(x,y) + 0 \Longrightarrow H(x,y) + g(x)$ an y-anti-derivative

Exact Equations

y solves $M(x,y)dx + N(x,y)dy = 0 \Longrightarrow y$ solves $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

Assume $f(x,y)$ w/

$$f_x(x,y) = M(x,y)$$
$$f_y(x,y) = N(x,y)$$

$\downarrow$

Recall multivariable chain rule:

$$\frac{d}{dt} f(\phi(t), \psi(t)) = f_x(\phi(t), \psi(t))\phi'(t) + f_y(\phi(t), \psi(t))\psi'(t)$$

$\downarrow$

$$\frac{d}{dx} f(x, y(x)) = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$
$$= M(x,y)\cdot 1 + N(x,y)\frac{dy}{dx}$$
$$= 0$$

since y solves $M(x,y)dx + N(x,y)dy = 0$.

Thus $f(x, y(x)) = c$ is an implicit equation for the solution $y = y(x)$.

Exact Equation: $M(x,y)dx + N(x,y)dy = 0$ such that there is an $f(x,y)$ with

$$f_x(x,y) = M(x,y)$$
$$f_y(x,y) = N(x,y).$$

Theorem

i) If $M(x,y)dx + N(x,y)dy = 0$ is exact with $f_x(x,y) = M(x,y)$, $f_y(x,y) = N(x,y)$, then $f(x,y) = c$ is an implicit solution

ii) $M(x,y)dx + N(x,y)dy = 0$ exact iff $M_y = N_x$ (and M and N are nice enough)

How to solve exact Equation

1) check exactness: $M_y = N_x$. If exact, proceed...

2) Exact $\Rightarrow$ $f_x = M$, $f_y = N$ for some $f(x,y)$.

3) Partial integration:

$$f_x = M \longrightarrow f(x,y) = \int M(x,y)\,dx + g(y) \quad \text{for some } g(y)$$
$$f_y = N \longrightarrow f(x,y) = \int N(x,y)\,dy + h(x) \quad \text{for some } h(x)$$

4) Use

$$\int M(x,y)\,dx + g(y) = f(x,y) = \int N(x,y)\,dy + h(x)$$

to solve for $g(y)$ or $h(x)$, and then for $f(x,y)$

5) Implicit solution is $f(x,y) = c$.

Ex Solve $2xy\,dx + (x^2-1)\,dy = 0$

1) $M(x,y) = 2xy$, $N(x,y) = x^2-1$

$M_y = 2x = N_x \Rightarrow$ ODE is exact

2) Let $f(x,y)$ be such that

$$f_x(x,y) = M(x,y) = 2xy$$
$$f_y(x,y) = N(x,y) = x^2-1$$

3) Integrate

$$f(x,y) = \int f_x(x,y)\,dx + g(y) = \int 2xy\,dx + g(y) = x^2y + g(y)$$

and

$$f(x,y) = \int f_y(x,y)\,dy + h(x) = \int x^2 - 1\,dy + h(x) = x^2y - y + h(x).$$

4) By 3):

$$f(x,y) = x^2y + g(y) = x^2y - y + h(x).$$

Thus

$$g(y) = -y \quad (\text{or } h(x) = 0),$$

and so

$$f(x,y) = x^2y + g(y) = x^2y - y \quad \longleftarrow \text{not the solution!}$$

5) Implicit solution is given by

$$f(x,y) = x^2y - y = c.$$

Ex Solve $(e^{2y}-y\cos xy)dx+(2xe^{2y}-x\cos xy+2y)dy=0$

1) $M(x,y)=e^{2y}-y\cos xy,\ N(x,y)=2xe^{2y}-x\cos xy+2y$

$$\left.\begin{array}{l} M_y=2e^{2y}-\cos xy+xy\cos xy \\ N_x=2e^{2y}-\cos xy+xy\sin xy\end{array}\right\} \to M_y=N_x \implies \text{exact.}$$

2) $f_x(x,y)=M(x,y)=e^{2y}-y\cos xy$

$f_y(x,y)=N(x,y)=2xe^{2y}-x\cos xy+2y$

3) $f(x,y)=\int f_x(x,y)\,dx+g(y)=\int e^{2y}-y\cos xy\,d\ +g(y)=xe^{2y}-\sin xy+g(y)$

$f(x,y)=\int f_y(x,y)dy+h(x)=\int 2xe^{2y}-x\cos xy+2y\,dy+h(x)=xe^{2y}-\sin xy+y^2+h(x)$

4) $f(x,y)=xe^{2y}-\sin xy+g(y)=xe^{2y}-\sin xy+y^2+h(x)$

$\hookrightarrow$ $g(y)=y^2$ (or $h(x)=0$)

$\hookrightarrow$ $f(x,y)=xe^{2y}-\sin xy+g(y)=xe^{2y}-\sin xy+y^2$ $\leftarrow$ not the solution !!

5) Solution is $f(x,y)=xe^{2y}-\sin xy+y^2=C.$

Ex Solve the IVP

$$\begin{cases} y'=\dfrac{xy^2-\cos x\sin x}{y(1-x^2)} \\ y(0)=2\end{cases}$$

1) $y'=\dfrac{xy^2-\cos x\sin x}{y(1-x^2)} \iff y(1-x^2)dy=(xy^2-\cos x\sin x)dx$

$\iff -(xy^2-\cos x\sin x)dx+y(1-x^2)dy$

$M(x,y)=-(xy^2-\cos x\sin x),\ N(x,y)=y(1-x^2)$

$$\left.\begin{array}{l} M_y=-2xy \\ N_x=-2xy\end{array}\right\} \to M_y=N_x \implies \text{exact}$$

2) $f_x=-(xy^2-\cos x\sin x)$

$f_y=y(1-x^2)$

3) $f(x,y)=\int -xy^2+\cos x\sin x\,dx+g(y)=-\frac{1}{2}x^2y^2-\frac{1}{2}\cos^2x+g(y)$

$f(x,y)=\int y(1-x^2)dy+h(x)=\frac{1}{2}y^2(1-x^2)+h(x)=\frac{1}{2}y^2-\frac{1}{2}x^2y^2+h(x)$

4) $f(x,y)=-\frac{1}{2}x^2y^2-\frac{1}{2}\cos^2x+g(y)=\frac{1}{2}y^2-\frac{1}{2}x^2y^2+h(x)$

$\hookrightarrow$ $g(y)=\frac{1}{2}y^2$ (or $h(x)=-\frac{1}{2}\cos^2x$)

$\hookrightarrow$ $f(x,y)=-\frac{1}{2}x^2y^2-\frac{1}{2}\cos^2x+\frac{1}{2}y^2$ $\longleftarrow$ not the solution !!!

5) Solution is $f(x,y)=-\frac{1}{2}x^2y^2-\frac{1}{2}\cos^2x+\frac{1}{2}y^2=C.$

Now we need to find C such that $y(0)=2$. So we plug in $(x,y)=(0,2)$ into $f(x,y)$:

$C=f(0,2)=-\frac{1}{2}0^2\cdot 2^2-\frac{1}{2}\cos^2 0+\frac{1}{2}2^2=0-\frac{1}{2}+2=\frac{3}{2}$

Thus

$$-\frac{1}{2}x^2y^2 - \frac{1}{2}\cos^2 x + \frac{1}{2}y^2 = \frac{3}{2}$$

solves the given IVP