2.3 Linear Equations

Main focus: Analytically solve first order ODE of the form

$$a_1(x)\, y' + a_0(x)\, y = g(x)$$

Observe that this ODE is equivalent to

$$y' + P(x)\, y = f(x) \quad \leftarrow \text{called standard form}$$

$$P(x) = \frac{a_0(x)}{a_1(x)}$$

$$f(x) = g(x)/a_1(x)$$

everywhere $a_1(x) \neq 0$. All x with $a_1(x) = 0$ are called singular points.

Observation, if we multiply

$$y' + P(x)\, y = f(x)$$

by an auxiliary function $\mu(x)$:

$$\mu(x)\, y' + \mu(x) P(x)\, y = \mu(x) f(x),$$

and if $\mu(x)$ is such that

$$\frac{d}{dx}[\mu(x) y] = \mu(x)\, y' + \mu(x) P(x)\, y,$$

then $y' + P(x)y = f(x)$ can be solved by solving

$$\frac{d}{dx}[\mu(x) y] = \mu(x)\, y' + \mu(x) P(x)\, y = \mu(x) f(x).$$

This ODE may be solved by integration:

$$\mu(x)\, y = \int \frac{d}{dx}[\mu(x) y]\, dx = \int \mu(x) f(x)\, dx + c$$

so

$$y = \frac{1}{\mu(x)} \int \mu(x) f(x)\, dx + \frac{1}{\mu}\, c.$$

So what is $\mu(x)$? Recall that we wanted

$$\frac{d}{dx}[\mu(x) y] = \mu(x)\, y' + \mu(x) P(x)\, y.$$

Then by chain rule

$$\cancel{\mu(x) \frac{dy}{dx}} + \frac{d\mu}{dx}\, y = \cancel{\mu(x) \frac{dy}{dx}} + \mu(x) P(x)\, y$$

$$\Updownarrow$$

$$\frac{d\mu}{dx}\, y = \mu(x) P(x)\, y$$

$$\Updownarrow$$

$$\frac{d\mu}{dx} = \mu(x)P(x).$$

But this equation is separable, and so

$$\frac{1}{\mu}\,d\mu = P(x)\,dx$$
$$\Updownarrow$$
$$\int \frac{1}{\mu}\,d\mu = \int P(x)\,dx$$
$$\Updownarrow$$
$$\ln \mu = \int P(x)\,dx$$
$$\Updownarrow$$
$$\mu = e^{\int P(x)\,dx}.$$

← Called the integration factor.

Recap on how to solve $a_1(x)y' + a_0(x)y = g(x)$:

1) Place ODE into standard form

$$(*) \qquad y' + P(x)y = f(x)$$

noting where $a_1(x) = 0$.

2) Compute the integrating factor

$$\mu = e^{\int P(x)dx}$$

3) Multiply both sides of $(*)$ by μ and obtain

$$\mu f = \mu y' + \mu P y = \frac{d}{dx}[\mu y].$$

4) Integrate and solve for y:

$$\frac{d}{dx}[\mu y] = \mu f$$
$$\Updownarrow$$
$$\mu y = \int \mu f\,dx + c$$
$$\Updownarrow$$
$$y = \frac{1}{\mu}\int \mu f\,dx + \frac{1}{\mu}c$$
$$\Updownarrow$$
$$y = e^{-\int P dx}\int e^{\int P dx} f\,dx + e^{-\int P dx}c$$

General Solution on an interval I: a param. fam. of solution that gives every possible particular solution on I.

Actually the derivation above shows that if P and f are continuous on I, then

$$y = \frac{1}{\mu}\int \mu f \, dx + \frac{1}{\mu} c$$

is the general solution to $y' + Py = f$ on I.

If I contains any singular points, the analysis is a bit trickier.

Ex Solve $y' - 3y = 0$.

1) ODE is already in standard form: $P(x) = -3$, $f(x) = 0$. $a_1(x) = 1$ is never zero.

2) Integrating factor:

$$\int P(x)\,dx = \int -3\,dx = -3x$$

$$\mu = e^{\int P(x)dx} = e^{-3x}$$

3) Multiply by μ:

$$e^{-3x}y' - 3e^{-3x}y = e^{-3x}\cdot 0 \iff \frac{d}{dx}[e^{-3x}y] = 0.$$

4) Integrating and solving for y:

$$\int \frac{d}{dx}[e^{-3x}y]\,dx = \int 0\,dx \iff e^{-3x}y = c \iff y = ce^{3x}.$$

Therefore, the solution is given as $y = ce^{3x}$.

Note that since $a_1(x) = 1 \neq 0$ everywhere, each step is valid for $-\infty < x < \infty$, and so the largest interval $y = ce^{3x}$ is a solution is $(-\infty, \infty)$.

Ex Solve $y' - 3y = 6$.

1) ODE is in standard form: $P(x) = -3$, $f(x) = 6$. Note $a_1(x) = 1 \neq 0$ everywhere.

2) From previous example, $\mu = e^{-3x}$.

3) Multiplying by μ:

$$e^{-3x}y' - 3e^{-3x} = 6e^{-3x} \iff \frac{d}{dx}[e^{-3x}y] = 6e^{-3x}$$

4) Integrating and solving for y:

$$\int \frac{d}{dx}[e^{-3x}y]\,dx = \int 6e^{-3x}\,dx \iff e^{-3x}y = -2e^{-3x} + c$$

$$\iff y = -2 + ce^{3x}.$$

Again, since $a_1(x) \neq 0$ everywhere, we may conclude that the largest interval $y = -2 + Ce^{3x}$ is a solution is $(-\infty, \infty)$.

Ex Find general solution of $(x^2-9)y' + y = 0$. Find largest intervals for which the general solution can be defined.

1) Place in standard form: $a_1(x) = x^2 - 9$:

$$y' + \frac{1}{x^2-9} y = 0.$$

Note $P(x) = \frac{1}{x^2-9}$, $f(x) = 0$.

Note $a_1(x) = 0$ when $x = \pm 3$. So x can only be in $(-\infty,-3)$, $(-3,3)$ or $(3,+\infty)$. Both $(-\infty,-3)$ and $(3,+\infty)$ are larger that $(-3,3)$. So we assume x is in $(-\infty,-3)$ or $(3,+\infty)$.

2) Integrating factor:

$$\int P(x)\,dx = \int \frac{1}{x^2-9}\,dx = \int -\frac{1}{6}\frac{1}{(x+3)} + \frac{1}{6}\frac{1}{(x-3)}\,dx$$
$$= -\frac{1}{6} \ln|x+3| + \frac{1}{6}\log|x-3|$$
$$= \frac{1}{6} \ln\left|\frac{x-3}{x+3}\right|$$
$$= \ln\left|\frac{x-3}{x+3}\right|^{1/6}$$

$$\frac{1}{x^2-9} = \frac{A}{x+3} + \frac{B}{x-3} \rightarrow 1 = (x-3)A + (x+3)B$$
$$\rightarrow A = -\frac{1}{6},\ B = \frac{1}{6}$$

$$\mu = e^{\int P\,dx} = e^{\ln\left|\frac{x-3}{x+3}\right|^{1/6}} = \left|\frac{x-3}{x+3}\right|^{1/6}$$

3) Multiplying by μ:

$$\left|\frac{x-3}{x+3}\right|^{1/6} y + \left|\frac{x-3}{x+3}\right|^{1/6} \frac{1}{x^2-9} y = 0 \quad \leftrightarrow \quad \frac{d}{dx}\left[\left|\frac{x-3}{x+3}\right|^{1/6} y\right] = 0$$

4) Integrating and solving for y:

$$\left|\frac{x-3}{x+3}\right|^{1/6} y = \int \frac{d}{dx}\left[\left|\frac{x-3}{x+3}\right|^{1/6} y\right]dx = \int 0\,dx = C$$

So

$$y = \left|\frac{x+3}{x-3}\right|^{1/6}$$

Now of the three intervals $(-\infty,-3)$, $(-3,3)$, $(3,+\infty)$, the two $(-\infty,-3)$, $(3,+\infty)$ are the largest.

Now that we have had practice with the derivation, we may black box some steps.
The algorithm is now:

1) $a_1(x)y' + a_0(x)y = g(x) \longrightarrow y' + P(x)y = f(x)$, $P(x) = \frac{a_0(x)}{a_1(x)}$, $f(x) = \frac{g(x)}{a_1(x)}$, not where $a_1(x) = 0$.

2) $\mu = e^{\int P(x)dx}$

$$\Downarrow$$

$$\boxed{\text{Things just happen}}$$

$$\Downarrow$$

3) Write

$$(*) \quad \frac{d}{dx}[\mu y] = \mu f$$

4) Integrate $(*)$ and solve for y:

$$\mu y = \int \mu f\,dx + c \Rightarrow y = \frac{1}{\mu}\int \mu f\,dx + \frac{c}{\mu}.$$

Ex Solve the IVP

$$\begin{cases} x^2 y' + y = 1 \\ y(2) = 2 \end{cases}$$

I) Solve the ODE $x^2y' + y = 1$

II) Find c such that $y(2) = 2$.

I) 1) $x^2y' + y = 1 \Rightarrow y' + \frac{1}{x^2}y = \frac{1}{x^2}$, $P(x) = \frac{1}{x^2}$, $f(x) = \frac{1}{x^2}$, $a_1(x) = \frac{1}{x^2}$ which is zero at $x = 0$.
We must then ensure x is in $(-\infty, 0)$ or $(0, \infty)$.

2)

$$\int P(x)\,dx = \int \frac{1}{x^2}\,dx = -\frac{1}{x}$$

$$\mu = e^{\int P(x)dx} = e^{-\frac{1}{x}}.$$

3)

$$\frac{d}{dx}[\mu y] = \mu f \rightarrow \frac{d}{dx}[e^{-\frac{1}{x}}y] = e^{-\frac{1}{x}}\frac{1}{x^2}$$

4)

$$\begin{aligned} e^{-\frac{1}{x}}y &= \int e^{-\frac{1}{x}}\frac{1}{x^2}\,dx + C \\ &= \int e^u\,du + C \qquad (u = -\tfrac{1}{x},\ du = \tfrac{1}{x^2}dx) \\ &= e^u + C \\ &= e^{-\frac{1}{x}} + C \\ y &= 1 + Ce^{\frac{1}{x}} \end{aligned}$$

II) $2 = y(2) = 1 + ce^{\frac{1}{2}} \Rightarrow ce^{\frac{1}{2}} = 1 \Rightarrow c = e^{-\frac{1}{2}}$.

So

$$y = 1 + e^{-\frac{1}{2}}e^{\frac{1}{x}} = 1 + e^{\frac{1}{x} - \frac{1}{2}}$$

solves the given IVP.

Transient term: terms involving C that vanish at $+\infty$.

Ex The general solution to $y' + y = x$ is $y = x - 1 + ce^{-x}$.
Find transient term.

Only the ce^{-x} term involves C. Moreover, as $x \to +\infty$, $ce^{-x} \to 0$.
So ce^{-x} is the transient term.

Ex The general solution to $(x^2-4)y'+4y=(x+2)^2$ is $y=\frac{(x+2)(C+x)}{x-2}$.

Find the transient term.

We expand:

$$y=\frac{Cx+x^2+2C+2x}{x-2}=\frac{Cx}{x-2}+\frac{x^2}{x-2}+\frac{2C}{x-2}+\frac{2x}{x-2}$$

Terms involving C: $\frac{2C}{x-2}$ and $\frac{Cx}{x-2}$.

Note

$$\lim_{x\to+\infty}\frac{2C}{x-2}=0 \quad \text{and} \quad \lim_{x\to+\infty}\frac{Cx}{x-2}=C.$$

So the transient term is $\frac{2C}{x-2}$.