Math 2410 | # Practice Exam I | **Feb x^{th}, 2019**

Name: Exam One Key

Instructions:

- Answer each question to the best of your ability. Show your work or receive no credit.
- All answers must be written clearly. Be sure to erase or cross out any work that you do not want graded. Partial credit can not be awarded unless there is legible work to assess.
- If you require extra space for any answer, you may use the back sides of the exam pages. Please indicate when you have done this so that I do not miss any of your work.

Academic Integrity Agreement

I certify that all work given in this examination is my own and that, to my knowledge, has not been used by anyone besides myself to their personal advantage. Further, I assert that this examination was taken in accordance with the academic integrity policies of the University of Connecticut.

Signed: Exam One Key (full name)

Questions:	1	2	3	4	5	**Total**
Score:						

1. (5 points) Solve the linear initial value problem

$$\begin{cases}(x^2-1)\frac{dy}{dx}+2y=(x+1)^2\\ y(2)=0\end{cases}$$

Hint: Partial fractions may be helpful.

Standard form: $y' + \frac{2}{x^2-1}y = \frac{(x+1)^2}{x^2-1} = \frac{x+1}{x-1}$

$P(x) = \frac{2}{x^2-1}$ $\quad \int \frac{2}{x^2-1}dx = \int \frac{1}{x-1} - \frac{1}{x+1}\,dx = \ln(x-1) - \ln(x+1) = \ln\frac{x-1}{x+1}$

$f(x) = \frac{x+1}{x-1}$ $\quad \mu = e^{\int p(x)dx} = \frac{x-1}{x+1}$

$\therefore \quad y = \frac{1}{\mu}\int \mu P\,dx + \frac{c}{\mu} = \frac{x+1}{x-1}\int \frac{x-1}{x+1}\cdot\frac{x+1}{x-1}\,dx + C\frac{x+1}{x-1}$

$= x\frac{x+1}{x-1} + C\frac{x+1}{x-1}$

$0 = y(2) = 2\cdot\frac{3}{1} + C\frac{3}{1} \to \quad 0 = 6 + C3 \to \quad C = -2$

$\therefore \quad y = x\frac{x+1}{x-1} - 2\frac{x+1}{x-1}$

2. (5 points) Consider the ODE

$$(y^3 + kxy^4 - 2x)dx + (3xy^2 + 20x^2y^3)dy = 0,$$

where k is to be determined.

(a) Find k so that the given ODE is an exact equation.

(b) Using this value of k, solve the given ODE.

a) $M(x,y) = y^3 + kxy^4 - 2x \rightarrow M_y = 3y^2 + 4kxy^3$

$N(x,y) = 3xy^2 + 20x^2y^3 \rightarrow N_x = 3y^2 + 40xy^3$

Exact $\longleftrightarrow M_y = N_x \longleftrightarrow \cancel{3y^2} + 4kxy^3 = \cancel{3y^2} + 40xy^3$

$\longleftrightarrow 4kxy^3 = 40xy^3$

$\longleftrightarrow k = 10$

b) $(y^3 + 10xy^4 - 2x)dx + (3xy^2 + 20x^2y^3)dy = 0$

$f_x = y^3 + 10xy^4 - 2x \rightarrow f = xy^3 + 5x^2y^4 - x^2 + g(y)$

$f_y = 3xy^2 + 20x^2y^3 \rightarrow f = xy^3 + 5x^2y^4 + h(x)$

$\therefore g(y) = 0, h(x) = -x^2$

$f(x,y) = xy^3 + 5x^2y^4 - x^2 = c$

3. (5 points) Consider the ODE
$$(x^k + ye^{y/x})dx - xe^{y/x}dy = 0,$$
where k is to be determined.

(a) Find k so that the given ODE is a homogeneous equation.

(b) Using this value of k, solve the given ODE.

a) $M(x,y) = x^k + y e^{y/x}$

$M(tx, ty) = t^k x^k + t y e^{y/x} = t(t^{k-1} x^k + y e^{y/x}) \to k=1$

$N(x,y) = -x e^{y/x}$

$N(tx, ty) = -t x e^{y/x} = t N(x,y)$

So hom. if $k=1$

b) Use $y = xu$, $dy = x du + u dx$

$(x + y e^{y/x}) dx - x e^{y/x} dy = 0 \longleftrightarrow (x + ux e^u) dx - x e^u (x du + u dx) = 0$

$\longleftrightarrow x dx + \cancel{ux e^u} dx - x^2 e^u du - \cancel{xu e^u} dx = 0$

$\longleftrightarrow x dx - x^2 e^u du = 0$

$\longleftrightarrow \frac{1}{x} dx = e^u du$

$\longleftrightarrow \ln x = e^u + C$

$\overset{u = y/x}{\longleftrightarrow} \ln x = e^{y/x} + C$

4. (5 points) Consider the Bernoulli equation

$$y' - y = y^p,$$

where $0 < p < 1$ is arbitrary.

(a) Solve the given equation with p left arbitrary. The solution will depend on p.

(b) Let y_p denote your answer in part (a). Compute

$$\lim_{p \to 0} y_p$$

and compare the limit to the solution to $y' - y = 1$.

(c) (If you have too much time on your hands) What happens as $p \to 1$? One should expect the limit is the solution to $y' - y = y$... But the limit isn't even well-defined for all x...

a) Recall the substitution should be $y = u^{\frac{1}{1-p}}$

$\therefore \frac{dy}{dx} = \frac{1}{1-p} u^{\frac{1}{1-p}-1} \frac{du}{dx}$

$\frac{dy}{dx} - y = y^p \rightarrow \frac{1}{1-p} u^{\frac{1}{1-p}-1} \frac{du}{dx} - u^{\frac{1}{1-p}} = u^{\frac{p}{1-p}}$

$\rightarrow \frac{du}{dx} - (1-p)u = 1-p$

Linear Eq:

$P(x) = -(1-p) = p-1$

$f(x) = 1-p$

$\mapsto \mu = e^{\int (p-1)dx} = e^{(p-1)x}$

$\therefore u = \frac{1}{\mu} \int \mu f \, dx + \frac{C}{\mu}$

$= e^{(1-p)x} \int (1-p) e^{(p-1)x} dx + Ce^{(1-p)x}$

$= -1 + Ce^{(1-p)x}$

$\therefore y = u^{\frac{1}{1-p}} = \left(-1 + Ce^{(1-p)x}\right)^{\frac{1}{1-p}}$

b) $\lim_{p \to 0^+} (-1 + Ce^{(1-p)x})^{\frac{1}{1-p}} = (-1 + Ce^{(1-0)x})^{\frac{1}{1-0}} = -1 + Ce^{x}$

$y' - y = 1 \rightarrow \mu = e^{\int -1 dx} = e^{-x}$, $y = \frac{1}{\mu} \int \mu f \, dx + \frac{C}{\mu} = e^{x} \int e^{-x} dx + Ce^{x} = -1 + Ce^{x}$

Actually, well-definedness depends on x and C...

5. (5 points) Consider the IVP

$$\begin{cases} y' = x + y^2 \\ y(0) = 0. \end{cases}$$

Let y be a solution to this IVP, and use Euler's method with step $h = 0.1$ to approximate $y(0.3)$. Indicate clearly all relevant $x_k, y_k, f(x_k, y_k)$. Round your final answer to 3 decimal places.

$h = 0.1, \; x_0 = 0, y_0 = 0, \; f(x,y) = x + y^2$

$y_1 = f(x_0, y_0) \cdot h + y_0$

$= f(0,0) \cdot 0.1 + 0 = 0$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$y_2 = f(0.1, 0) \cdot 0.1 + 0$

$= (0.1 + 0^2) 0.1 + 0$

$= 0.01$

$x_2 = 0.1 + 0.1 = 0.2$

$y_3 = f(0.2, 0.01) \cdot 0.1 + 0.01$

$= 0.03001$

$x_3 = 0.2 + 0.1 = 0.3$

$\therefore \; y(0.3) = y(x_3) \approx y_3 = 0.03001$

≈ 0.030