2.2 Separable Equations

First analytic technique for solving a certain class of ODEs.

Idea: some ODEs take the form

$$\frac{dy}{dx} = f(x)g(y). \quad \left(\text{e.g., } \frac{dy}{dx} = x^2 \cos y\right)$$

This may rewritten as

$$\frac{1}{g(y)}dy = f(x)dx \quad (\text{Note we implicitly assume } g(y) \neq 0.)$$

This suggests we can integrate:

$$\int \frac{1}{g(y)}dy = \int f(x)dx.$$

Either
a) we can then algebraically solve for y, or
b) this will implicitly define y as a function of x.

Note: this is all formal. One can justify everything – see textbook for justification

We proceed by example (on WebAssign, you may need other calc 2 techniques)

Ex
Solve $y' = \frac{x}{y}$.

Solution:
$$\frac{dy}{dx} = \frac{x}{y} \Rightarrow y\,dy = x\,dx \Rightarrow \int y\,dy = \int x\,dx \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$
Solve for y:
$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C \Rightarrow y^2 = x^2 + 2C \Rightarrow y = \pm\sqrt{x^2 + 2C}$$
We can relabel the constant $2C$ by C to obtain (since C was arbitrary anyways)
$$y = \pm\sqrt{x^2 + C}.$$

Ex Solve the IVP
$$dy - y^2 dx = 0$$
$$y(0) = 2$$

Solution:
$$dy - y^2 dx = 0 \Rightarrow dy = y^2 dx \Rightarrow \frac{1}{y^2}dy = dx \Rightarrow \int \frac{1}{y^2}dy = \int dx \Rightarrow -y^{-1} = x + C.$$
Solving for y:
$$-y^{-1} = x + C \Rightarrow y = -\frac{1}{x + C}$$
Initial condition $y(0) = 2$ gives:
$$2 = y(0) = -\frac{1}{0 + C} = -\frac{1}{C} \Rightarrow C = -\frac{1}{2}$$
Therefore $y = -\frac{1}{x - \frac{1}{2}}$ is the solution to the IVP.

Ex Solve the ODE: $(e^{2y}-y)y' = e^{x}$

Solution:

$$(e^{2y}-y)\frac{dy}{dx} = e^{x} \Rightarrow (e^{2y}-y)dy = e^{x}dx \Rightarrow \int e^{2y}-y\,dy = \int e^{x}dx \Rightarrow \frac{1}{2}e^{2y}-\frac{1}{2}y^2 = e^{x}+c.$$

Therefore we arrive at the expression

$$\frac{1}{2}e^{2y}-\frac{1}{2}y^2 = e^{x}+c\ldots$$

There is no hope in solving for y... This is an example of the solution to an ODE being implicitly defined; i.e.,

$$\frac{1}{2}e^{2y}-\frac{1}{2}y^2 = e^{x}+c$$

is called an implicit solution.

Ex Solve the ODE

$$\frac{dy}{dx} = \frac{xy+10y-x-10}{xy-11y+x-11}.$$

Problem is recognizing this is indeed separable. After some thinking, one sees

$$xy+10y-x-10 = (x+10)(y-1)$$

$$xy-11y+x-11 = (x-11)(y+1).$$

Then

$$\frac{dy}{dx} = \frac{(x+10)(y-1)}{(x-11)(y+1)} \Rightarrow \frac{y+1}{y-1}dy = \frac{x+10}{x-11}dx.$$

We compute

$$\int \frac{y+1}{y-1}dy = \int \frac{y-1+2}{y-1}dy$$

$$= \int 1+\frac{2}{y-1}dy$$

$$= y+2\ln|y-1|+C$$

$$\int \frac{x+10}{x-11}dx = \int \frac{x-11+21}{x-11}dx$$

$$= \int 1+\frac{21}{x-11}dx$$

$$= x+21\ln|x-11|+C.$$

Therefore, our (implicit) solution is

$$y+2\ln|y-1| = x+21\ln|x-11|+c.$$