1.3 Differential Equations as Mathematical Models

Idea: Models of time evolution / rates of change can be modeled with DEs.

Ex Population dynamics:

Rate of population growth proportional to current population

$$\Updownarrow$$

$$\begin{pmatrix}\text{Rate of pop.} \\ \text{growth}\end{pmatrix} = k \left(\text{pop.}\right)$$

$$\frac{dP}{dt} = k\,P$$

Example 1

$x(t)$ = infected

rate of infection = rate of disease spread = $\frac{dx}{dt}$

$n - x(t)$ = healthy people

Assume each sick person interacts w/ healthy person:

$$\#\text{ of interactions} = (\text{infected}) \times (\text{healthy})$$
$$= x(t)(n - x(t)).$$

$$\text{Rate of disease spread} = k\ \#\text{ of interactions}$$

$$\frac{dx}{dt} = k\,x(n-x)$$

Example 2

$A(t)$ = amount of salt in tank

$$\begin{pmatrix}\text{Rate at which} \\ \text{amount of salt} \\ \text{changes}\end{pmatrix} = \begin{pmatrix}\text{Input} \\ \text{Rate} \\ \text{of salt}\end{pmatrix} - \begin{pmatrix}\text{Output} \\ \text{Rate} \\ \text{of salt}\end{pmatrix}$$

$$\frac{dA}{dt} = R_{in} - R_{out}$$

$$\begin{pmatrix}\text{Input rate} \\ \text{of salt}\end{pmatrix} = \begin{pmatrix}\text{Concentration of} \\ \text{salt in inflow}\end{pmatrix} \times \begin{pmatrix}\text{input rate} \\ \text{of brine}\end{pmatrix}$$

$$R_{in} = 2\,\frac{\text{lb}}{\text{gal}} \times 3\,\frac{\text{gal}}{\text{min}}$$
$$= 6\ \text{lb/min}$$

$$\begin{pmatrix}\text{Output rate}\\ \text{of salt}\end{pmatrix} = \begin{pmatrix}\text{Concentration of}\\ \text{salt in outflow}\end{pmatrix} \times \begin{pmatrix}\text{output rate}\\ \text{of brine}\end{pmatrix}$$

$$= \frac{\text{Amount of salt}}{\text{Volume}} \times \begin{pmatrix}\text{output rate}\\ \text{of brine}\end{pmatrix}$$

$$R_{out} = \frac{A(t)}{300} \frac{\text{lb}}{\text{gal}} \times 3 \frac{\text{gal}}{\text{min}}$$

$$= \frac{A(t)}{100} \frac{\text{lb}}{\text{min}}$$

Therefore

$$\frac{dA}{dt} = R_{in} - R_{out} = 6 - \frac{A(t)}{100}.$$