1.2 Initial Value Problem

When modeling a system:

1) Know initial state of system
2) Have reasonable system evolution model
3) Want prediction capability

Ex Let $T(t)$ be temperature of object. T_a = ambient Temp.

1) $T(0) = T_0$ initial temp
2) $\frac{dT}{dt} = -k(T - T_a)$ Newton's Law of cooling
3) $T(t) = T_a + (T_0 - T_a)e^{-kt}$ - solution

Note: $T_0 > T_a \Rightarrow$ cooling
$T_0 < T_a \Rightarrow$ heating

Formalize as problem:

Solve $\frac{dT}{dt} = -k(T - T_a)$
subject to $T(0) = T_0$.

This is called an initial value problem.

In General:

Initial Value problem:

Solve $\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$

subject to initial conditions
$$y(x_0) = y_0$$
$$y'(x_0) = y_1$$
$$\vdots$$
$$y^{(n-1)}(x_0) = y_{n-1}$$

Note: Solving nth order is like "undoing" n derivatives. So we pick up n degrees of freedom. So we need n conditions.

Ex
$y = \frac{1}{1+ce^{-x}}$ is a 1-param. fam. sol. of $y' = y - y^2$. Solve the IVP

$$(*)\begin{cases} y' = y - y^2 \\ y(0) = 2 \end{cases}$$

$y(0) = 2 \Rightarrow 2 = y(0) = \frac{1}{1+c\cdot e^0} = \frac{1}{1+c} \Rightarrow 1 = 2(1+c) = 2 + 2c \Rightarrow 2c = -1$

So $c = -\frac{1}{2}$ and $y = \frac{1}{1-\frac{1}{2}e^{-x}}$ solves the IVP (*)

Ex

$y = c_1 \cos 4x + c_2 \sin 4x$ solves $y'' + 16y = 0$.

Solve the IVP

$$(**) \quad \begin{cases} y'' + 16y = 0 \\ y(\frac{\pi}{16}) = -\sqrt{2} \\ y'(\frac{\pi}{16}) = \sqrt{2}. \end{cases}$$

If $y = c_1 \cos 4x + c_2 \sin 4x$ is to solve $(**)$, then it must be true that

$$y(\tfrac{\pi}{16}) = -\sqrt{2} \quad \text{and} \quad y'(\tfrac{\pi}{16}) = \sqrt{2}$$

So

$$-\sqrt{2} = y(\tfrac{\pi}{16}) = c_1 \cos \tfrac{\pi}{4} + c_2 \sin \tfrac{\pi}{4} = \tfrac{\sqrt{2}}{2} c_1 + \tfrac{\sqrt{2}}{2} c_2$$

$$\sqrt{2} = y'(\tfrac{\pi}{16}) = -4c_1 \sin \tfrac{\pi}{4} + 4c_2 \cos \tfrac{\pi}{4} = -4c_1 \cdot \tfrac{\sqrt{2}}{2} + 4c_2 \tfrac{\sqrt{2}}{2}$$

Thus we must solve the system

$$\begin{aligned} \tfrac{\sqrt{2}}{2} c_1 + \tfrac{\sqrt{2}}{2} c_2 &= -\sqrt{2} \\ -2\sqrt{2} c_1 + 2\sqrt{2} c_2 &= \sqrt{2} \end{aligned} \iff \begin{aligned} c_1 + c_2 &= -2 \\ -c_1 + c_2 &= \tfrac{1}{2} \end{aligned}$$

$$\iff \begin{aligned} 2c_2 &= -\tfrac{3}{2} \\ 2c_1 &= -\tfrac{5}{2} \end{aligned}$$

$$\iff \begin{aligned} c_2 &= -\tfrac{3}{4} \\ c_1 &= -\tfrac{5}{4}. \end{aligned}$$

Therefore $y = -\frac{5}{4} \cos 4x - \frac{3}{4} \sin 4x$ solves the IVP $(**)$.