Chapter 1: Introduction to Differential Equations

1.1 Definitions and terminology

Q: What does it mean to solve

$$x^2-1=0$$

A: Find $x$ so that $x^2-1=0$ is true. E.g., $x=\pm 1$.

Q: What does it mean to solve

$$x^2+y^2=1$$

A: Find a function $f(x)$ or $g(y)$ so that

$$x^2+f(x)^2=1 \quad \text{or} \quad g(y)^2+y^2=1$$

is true. E.g., $f(x)=\sqrt{1-x^2}$, $f(x)=-\sqrt{1-x^2}$, $g(y)=\sqrt{1-y^2}$, $g(y)=-\sqrt{1-y^2}$

So an equation is a mathematical statement. Solving it is finding an appropriate mathematical object that satisfies the equation.

Q: What does it mean to solve

$$\frac{dy}{dx}=y$$

A: Find a function $y=f(x)$ so that

$$\frac{df}{dx}=f(x).$$

E.g., $f(x)=Ae^x$. Indeed,

$$f'(x)=\frac{d}{dx}(Ae^x)=Ae^x=f(x).$$

We shall study solving equations like the last question.

Differential Equation: an equation involving derivatives of one or more unknown functions with respect to one or more dependent variable.

Ex

a) $y'=y$

b) $\cos\left(\frac{df}{dx}\right)+\sin\left(\frac{dg}{dx}\right)=0$

c) $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$

**Ordinary Differential Equation**: DE w/ one independent variable.

**Order** of a DE: highest order derivative appearing in DE

Ex

a) $\underbrace{\frac{dy}{dx}}_{\text{order}=1} + y^2 = 0$

b) $\underbrace{y^{(3)}}_{\text{order}=3} + (y')^4 = x^3$

**Differential form**: $N(x,y)\,dy + M(x,y)\,dx = 0$.

This notation just stands for the ODE ("divide" by $dx$)

$$N(x,y)\frac{dy}{dx} + M(x,y) = 0.$$

Ex

a) $\cos x \frac{dy}{dx} + e^x = 0 \Rightarrow \cos x\,dy + e^x\,dx = 0$

b) $e^{xy}dy + x^2 dx = 0 \Rightarrow e^{xy}\frac{dy}{dx} + x^2 = 0.$

**Linear ODE**: Compare

$y = x^2+1$ and $y^2 = x^2+1$

power $=1 \Rightarrow$ linear in $y$ power $=2 \Rightarrow$ nonlinear in $y$

$y'' + y = \cos x$ and $(y')^2 + \cos y = x$

powers $=1 \Rightarrow$ linear in $y''$ and $y$ (power $=2$ not polynomial) $\Rightarrow$ nonlinear in $y'$ and $y$.

General: An ODE linear in each derivative is called a linear ODE:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y + g(x) = 0$$

**Solution to an ODE**: A function that satisfies the ODE on an interval $I$, where $I$ is called an **interval of definition**.

Ex verify $f(x) = -\frac{1}{\ln x}$ solves the ODE $xy' = y^2$ on $(0,\infty)$.

We set $y = f(x)$. Then $y' = \frac{1}{x}\left(\frac{1}{\ln x}\right)^2$ and

$$xy' = x\cdot\frac{1}{x}\left(\frac{1}{\ln x}\right)^2 = \left(\frac{1}{\ln x}\right)^2 = y^2 \text{ for all } x \text{ in } (0,\infty).$$

Note that $I = (0,\infty)$ cannot be any larger since $f(x)$ is undefined at $x=0$.

Ex Verify $f(x)=\frac{1}{x}$ solves the ODE $xy'+y=0$ on $(-\infty,0)$ and $(0,+\infty)$.

We set $y=f(x)$. Then $y'=-\frac{1}{x^2}$ and

$$xy'+y = x\cdot\left(-\frac{1}{x^2}\right)+\frac{1}{x} = -\frac{1}{x}+\frac{1}{x}=0 \text{ for all } x\neq 0.$$

Note that the domain of $f(x)=\frac{1}{x}$ is $(-\infty,0)\cup(0,\infty)$. The intervals $I=(-\infty,0)$ or $I=(0,+\infty)$ are appropriate intervals for $f$ as a solution.

Trivial solution: if $y=0$ solves an ODE, it is called the trivial solution.

Ex

a) $f(x)=0$ solves $y'=y$. Indeed, if $y=f$, then $y'=0=y$.

b) $f(x)=0$ does not solve $y'=y+1$. Indeed, if $y=f$, then $y'=0$, $y+1=0+1=1\neq y'$.

Parameterized family of solutions:

Observe $y=ce^x$, $c$ any real number, solves $y'=y$:

$$y=ce^x \Rightarrow y'=ce^x=y.$$

Thus $c$ parameterize the one parameter family of solutions $y=ce^x$.

Observe $y=c_1e^{-2x}+c_2e^{-x}$, $c_1$ and $c_2$ any real number, solves $y''+3y'+2y=0$.
Indeed,

$$y=c_1e^{-2x}+c_2e^{-x}$$
$$y'=-2c_1e^{-2x}-c_2e^{-x}$$
$$y''=4c_1e^{-2x}+c_2e^{-x}$$

$$y''+3y'+2y=4c_1e^{-2x}+c_2e^{-x}-6c_1e^{-2x}-3c_2e^{-x}+2c_1e^{-2x}+2c_2e^{-x}$$
$$=0 \checkmark$$

Thus $c_1$ and $c_2$ parameterize a two parameter family of solution

$$=c_1e^{-2x}+c_2e^{-x}.$$

More generally, an $n$th order ODE may have an $n$-parameter family of solutions.

Particular Solution: a solution free of parameters.

Ex

$y=c_1e^{-2x}+c_2e^{-x}$ was an 2-param. fam. of sol. from previous Ex. Then $y=2e^{-2x}$, $y=e^{-2x}+3e^{-x}$, etc., are particular solutions.