Problem 1

Follows by explicit computation:

$$X(t) = \begin{pmatrix}1\\1\end{pmatrix}e^{-3t} \longrightarrow X'(t) = -\begin{pmatrix}3\\3\end{pmatrix}e^{-3t}$$

$$AX(t) = \begin{pmatrix}3\\3\end{pmatrix}e^{-3t}$$

So $X'(t) \neq AX(t) \longrightarrow X(t)$ not a solution.

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Problem 2

$$\det\begin{pmatrix}1-\lambda & 0 & 1\\ 0 & 1-\lambda & 0\\ 1 & 0 & 1-\lambda\end{pmatrix} = (1-\lambda)^3 - (1-\lambda) \overset{\text{By hint}}{=} -\lambda^3+3\lambda^2-2\lambda = -\lambda(\lambda^2-3\lambda+2) = -\lambda(\lambda-1)(\lambda-2)$$

So $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 0$ are the eigenvalues.

Find eigenvector $K$ for $\lambda_3 = 0$:

$$\text{rref}\left(\begin{array}{ccc|c}1&0&1&0\\0&1&0&0\\1&0&1&0\end{array}\right) = \left(\begin{array}{ccc|c}1&0&1&0\\0&1&0&0\\0&0&0&0\end{array}\right) \longrightarrow k_1+k_3=0,\ k_2=0 \longrightarrow K = \begin{pmatrix}-k_3\\0\\k_3\end{pmatrix} = k_3\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

So general sol:

$$X(t) = c_1\begin{pmatrix}-1\\0\\1\end{pmatrix} + c_2\begin{pmatrix}1\\0\\1\end{pmatrix}e^{2t} + c_3\begin{pmatrix}0\\1\\0\end{pmatrix}e^{t}$$

Problem 3 Novel

(a) $W(X_1,X_2) = W(X_2,X_3) = 0 \longrightarrow X_1 = c_1X_2,\ X_2 = c_2X_3$ for some $c_1, c_2 \neq 0$.

Hence $X_1 = c_1c_2X_3$ $c_1c_2 \neq 0 \Rightarrow X_1$ and $X_2$ are linearly dependent

Hence $W(X_1,X_3) = 0 \Rightarrow X_1, X_3$ cannot form a fundamental set.

(b) Written Assignment

$$W(X_1(0), X_2(0), X_3(0)) = \det\begin{pmatrix}1&0&2\\2&2&3\\0&1&0\end{pmatrix} = 0+0+4-0-3-0 = 1 \neq 0$$

$$\begin{matrix}1&0&2&1&0\\2&2&3&2&2\\0&1&0&0&1\end{matrix}$$

So $X_1, X_2, X_3$ are linearly indep $\rightarrow$ they form a fundamental set.

Problem 4 WebAssign

Find Eigenvalues:

$$\det\begin{pmatrix}20-\lambda & -25\\ 4 & -\lambda\end{pmatrix} = -\lambda(20-\lambda)+100 = -20\lambda+\lambda^2+100 = (\lambda-10)^2$$

$\rightarrow$ eigenvalue $= \lambda = 10$ is repeated.

Should show only one eigenvector up to scaling.

General solution:

$$X(t) = V_0e^{\lambda t} + V_1te^{\lambda t}$$

$$V_0 = \begin{pmatrix}x_0\\y_0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$V_1 = (A-\lambda I)V_0 = \begin{pmatrix}10 & -25\\ 4 & -10\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}10\\4\end{pmatrix}$$

So $$X(t) = \begin{pmatrix}1\\0\end{pmatrix}e^{10t} + \begin{pmatrix}10\\4\end{pmatrix}te^{10t}$$

## Problem 5 WebAssign

$A = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$

Find eigenvalues:

$$\det\begin{pmatrix} 1-\lambda & 1 \\ -2 & -1-\lambda \end{pmatrix} = (1-\lambda)(-1-\lambda)+2 = -1-\lambda+\lambda+\lambda^2+2 = \lambda^2+1$$

So $\lambda_1 = i$, $\lambda_2 = -i$.

Eigenvector $K$ for $\lambda_1 = i$:

$$\left(\begin{array}{cc|c} 1-i & 1 & 0 \\ -2 & -1-i & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1-i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \to (1-i)k_1 + k_2 = 0 \to k_2 = (i-1)k_1 \to K = \begin{pmatrix} k_1 \\ (i-1)k_1 \end{pmatrix} = k_1\begin{pmatrix} 1 \\ i-1 \end{pmatrix}$$

So

$$\operatorname{Re} K = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \operatorname{Im} K = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

General Sol:

$$X(t) = c_1\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t\right] + c_2\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\sin t\right]$$

From class

## Problem 6

(a) $X(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-t}$ – straight line in direction of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(b) $X(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{t}$ – straight line in direction of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(c) $c_1 = c_2 = 1$:

$X(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-t}$

$X(+\infty) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cdot +\infty$

$X(-\infty) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cdot +\infty$

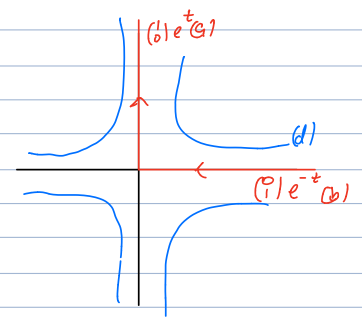
$c_1 = 1$, $c_2 = -1$

$X(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{t} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-t}$

$X(+\infty) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cdot +\infty$

$X(-\infty) = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cdot +\infty$.

(d)



$$\varphi_j(x) = (p(x), \alpha(\sigma_j(p(x)), x))$$

$$\psi_j(b,g) = \sigma_j(b)g$$

$$\varphi_j(\psi_j(b,g)) = \varphi_j(\sigma_j(b)g) = (p(\sigma_j(b)g), \alpha(\sigma_j(p(\sigma_j(b)g)), \sigma_j(b)g))$$

$$= (b, \alpha(\sigma_j(b), \sigma_j(b)g))$$

$$= (b,g)$$

$$\psi_j(\varphi_j(x)) = \psi_j(p(x), \alpha(\sigma_j(p(x)), x))$$

$$= \sigma_j(p(x))\alpha(\sigma_j(p(x)), x)$$

$$= x$$

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\pi_X"'] & & Y \arrow[dl, "\pi_Y"] \\ & M & \end{tikzcd}$$

$\pi_Y \circ \varphi = \pi_X$

$\pi_X^{-1}(x) = \varphi^{-1} \circ \pi_Y^{-1}(x)$

$\hookrightarrow \varphi(\pi_X^{-1}(x)) = \pi_Y^{-1}(x)$

$\hookrightarrow \varphi$ carries fibres of $X$ to $Y$.

$f$ equivariant

$$\begin{tikzcd} X_1 \arrow[d, "f"] \\ X_2 \end{tikzcd} \xrightarrow{\mathcal{P}} \begin{tikzcd} X_1 \arrow[r, "\mathcal{P}(f)"] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\ \tilde{X}_1 \arrow[r, "\exists!\, g"] & \tilde{X}_2 \end{tikzcd}$$

$$\pi_2 \circ \mathcal{P}(f) = g \circ \pi_1$$

$$\mathcal{P}(f) = f$$

$$\int_{\mathbb{B}^4} (-\Delta_{\mathbb{H}} - 2)(-\Delta_{\mathbb{H}})u \cdot u \, dV_{\mathbb{H}} = \int_{\mathbb{B}^4} [(-\Delta_{\mathbb{H}})(-\Delta_{\mathbb{H}})u - 2(-\Delta_{\mathbb{H}})u]u \, dV_{\mathbb{H}} = \int_{\mathbb{B}^4} [(-\Delta_{\mathbb{H}})\{(-\Delta_{\mathbb{H}})u - 2u\}]u \, dV_{\mathbb{H}}$$

$$= \int_{\mathbb{B}^4} (-\Delta_{\mathbb{H}})(-\Delta_{\mathbb{H}} - 2)u \cdot u \, dV_{\mathbb{H}} = \int_{\mathbb{B}^4} \left(\frac{1-|x|^2}{2}\right)^4 \Delta^2 u \cdot u \, dV_{\mathbb{H}} = \int_{\mathbb{B}^4} |\Delta u|^2 dz$$