Chapter 7 The Laplace Transform

Section 7.1: Def. of the Laplace Transform

Transforms:

- We understand the action
$$f(x) \xrightarrow{\frac{d}{dx}} f'(x)$$
as an "operation" on the space of differentiable functions.
- Here $\frac{d}{dx} : f(x) \mapsto f'(x)$ is called a differential transform
- Similarly, $\int_a : f(t) \mapsto \int_a^x f(t)dt$ is an integral transform.

General integral transforms

- Suppose $f(t)$, $K(s,t)$ are (square) integrable functions wrt t.
- We may define the integral transformation

$$\underset{\text{function of } t}{f(t)} \longrightarrow \underset{\text{function of } s}{F(s)} = \int_a^b K(s,t) f(t) dt.$$

- We call $K(s,t)$ the kernel of the transformation $f(t) \mapsto F(s)$.

(E.g., choosing K appropriately, we may define the Fourier transform, mollifiers, bandpass filters, etc.)

- We define
$$\int_a^\infty K(s,t) f(t) dt = \lim_{b \to +\infty} \int_a^b K(s,t) f(t) dt.$$

The Laplace Transform

- We consider the kernel $K(s,t) = e^{-st}$.
- Note that if $\Delta = \frac{d^2}{dt^2}$, then $\Delta K(s,t) = s^2 e^{-st}$, and so e^{-st} are eigenfunctions of the differential operator Δ.
- Define the transform
$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t) e^{-st} dt = \lim_{b \to +\infty} \int_0^b f(t) e^{-st}$$

provided the integral exists.

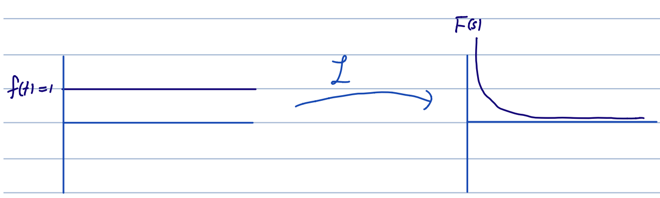
- We call this the Laplace Transform.

(Replacing s by is, we get the Fourier transform.)

Example

Evaluate $\mathcal{L}\{1\}$

$$\mathcal{L}\{1\} = \int_0^\infty 1 \cdot e^{-st} dt = \lim_{b \to +\infty} \int_0^b e^{-st} dt = \lim_{b \to +\infty} \frac{-e^{-sb}+1}{s} = \frac{1}{s}$$



Example

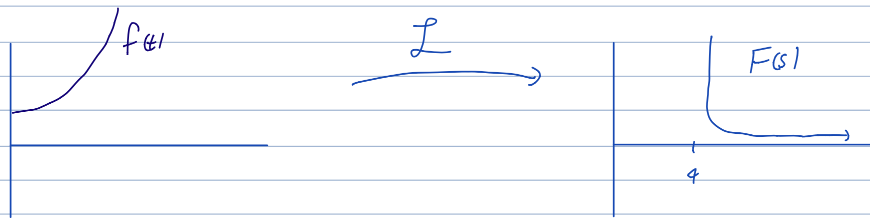
Evaluate $\mathcal{L}\{t\}$

$$\mathcal{L}\{t\} = \int_0^\infty t e^{-st} dt = \frac{-t e^{-st}}{s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} dt = \frac{1}{s}\mathcal{L}\{1\} = \frac{1}{s^2}$$

Example

Evaluate $\mathcal{L}\{e^{4t}\}$

$$\mathcal{L}\{e^{4t}\} = \int_0^\infty e^{4t} e^{-st}\,dt = \int_0^\infty e^{(4-s)t}\,dt = \lim_{b\to+\infty} \frac{1}{4-s}\left(e^{(4-s)b} - 1\right) = \frac{1}{s-4} \text{ provided } 4-s<0; \text{ i.e., } 4<s.$$



Example

Evaluate $\mathcal{L}\{e^{-t}\sin 2t\}$

$$\mathcal{L}\{e^{-t}\sin 2t\} = \int_0^\infty e^{-t}\sin 2t\, e^{-st}\,dt = \int_0^\infty \sin 2t\, e^{-t(s+1)}\,dt$$

$$= -\cancel{\frac{\sin 2t\, e^{-t(s+1)}}{s+1}}^{\,0}\Big|_0^\infty + \frac{2}{s+1}\int_0^\infty \cos 2t\, e^{-t(s+1)}\,dt$$

$$= -\frac{2}{s+1}\,\frac{\cos 2t\, e^{-t(s+1)}}{s+1}\Big|_0^{+\infty} - \frac{4}{(s+1)^2}\int_0^\infty \sin 2t\, e^{-t(s+1)}\,dt$$

$$= \frac{2}{(s+1)^2} - \frac{4}{(s+1)^2}\mathcal{L}\{e^{-t}\sin 2t\}$$

Solving for $\mathcal{L}\{e^{-t}\sin 2t\}$:

$$\frac{(s+1)^2+4}{(s+1)^2}\mathcal{L}\{e^{-t}\sin 2t\} = \frac{2}{(s+1)^2} \longrightarrow \mathcal{L}\{e^{-t}\sin 2t\} = \frac{2}{(s+1)^2+4}$$

Example

Evaluate $\mathcal{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 0 & 0\le t\le 1 \\ t & t>1 \end{cases}.$$

We find

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt = \int_0^1 0\cdot e^{-st}\,dt + \int_1^\infty t e^{-st}\,dt$$

$$= -\frac{te^{-st}}{s}\Big|_1^\infty + \frac{1}{s}\int_1^\infty e^{-st}\,dt$$

$$= \frac{e^{-s}}{s} + \frac{1}{s^2}e^{-s}$$

$$= e^{-s}\frac{(s+1)}{s^2}$$

Theorem

$\mathcal{L}$ is a linear operator:

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \int_0^\infty (\alpha f(t) + \beta g(t))e^{-st}\,dt$$

$$= \alpha\int_0^\infty f(t)e^{-st}\,dt + \beta\int_0^\infty g(t)e^{-st}\,dt$$

$$= \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}.$$

Existence of $\mathcal{L}\{f(t)\}$:

- We say $f(t)$ is of exponential order C provided there are constants $M>0, T>0$ such that

$$|f(t)| \le Me^{Ct} \text{ for all } t>T.$$

So for large t, f is controlled by Me^{ct}.

Theorem

If f is piecewise continuous on $[0,\infty)$ and of exponential order, then $\mathcal{L}\{f(t)\}$ exists for $s>c$.
and piecewise bounded

Section 7.2 Inverse Transforms and Transforms of Derivatives

7.2.1 Inverse Transforms

Q: Can we undo the Laplace transform?

Idea: Given $F(s)$, find $f(t)$ using tables and algebra so that $F(s)=\mathcal{L}\{f(t)\}$. We write $\mathcal{L}^{-1}\{F(s)\}=f(t)$.

Note: $\mathcal{L}\{\alpha f(t)+\beta g(t)\}=\alpha\mathcal{L}\{f(t)\}+\beta\mathcal{L}\{g(t)\}=\alpha F(t)+\beta G(t)\longrightarrow\mathcal{L}^{-1}\{\alpha F(t)+\beta G(t)\}=\alpha f(t)+\beta g(t)\rightarrow\mathcal{L}^{-1}$ is linear.

Example

Evaluate $\mathcal{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\}$ using $\sin kt=\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}$, $\cos kt=\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$

Idea: transform $\frac{-2s+6}{s^2+4}$ into an expression we can use the identities on:

$$\frac{-2s+6}{s^2+4}=\frac{-2s}{s^2+4}+\frac{6}{s^2+4}=-2\frac{s}{s^2+2^2}+3\frac{2}{s^2+2^2}$$

so

$$\mathcal{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\}=-2\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\}+3\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}=-2\cos 2t+3\sin 2t$$

Note: Anything uglier, simplify via algebra and may need to use partial fractions (E.g. Example 3 in book).

Example

Find $\mathcal{L}^{-1}\left\{\frac{s^2+2}{s^3}\right\}$ by first finding $\mathcal{L}\{at^2+a\}$

We find $\mathcal{L}\{at^2+a\}=a\mathcal{L}\{t^2\}+a\mathcal{L}\{1\}=\frac{2a}{s^3}+\frac{a}{s}=\frac{2a+as^2}{s^3}=a\frac{s^2+2}{s^3}$

Hence $a\mathcal{L}^{-1}\left\{\frac{s^2+2}{s^3}\right\}=at^2+a\ \rightarrow\ \mathcal{L}^{-1}\left\{\frac{s^2+2}{s^3}\right\}=t^2+1.$

7.2.2 Transforms of Derivatives

Observe $\mathcal{L}\{f'(t)\}=\int_0^\infty f'(t)e^{-st}dt=f(t)e^{-st}\Big|_0^\infty+s\int_0^\infty f(t)e^{-st}dt=-f(0)+s\mathcal{L}\{f(t)\}$ (assuming $\lim_{t\to\infty}f(t)e^{-st}=0$).

Theorem

Suppose $f,\dots,f^{(n-1)}$ cont. on $[0,\infty)$ of exponential order. Then

$$\mathcal{L}\{f^{(n)}(t)\}=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\dots-f^{(n-1)}(0),\quad n\geq 1,$$

if it exists, where $\mathcal{L}\{f(t)\}=F(s)$.

Solving Linear ODEs:

- Consider the IVP

$$\sum_{k=0}^{n}a_k y^{(k)}=g(t).$$
$$y(0)=y_0,\ \dots,\ y^{(n-1)}(0)=y_{n-1}.$$

- By Linearity of $\mathcal{L}$, and Theorem, we have

$$\begin{aligned}\mathcal{L}\left\{\sum_{k=0}^{n}a_k y^{(k)}\right\}&=\sum_{k=0}^{n}a_k\mathcal{L}\{y^{(k)}\}\\&=\sum_{k=0}^{n}a_k[s^kY(s)-s^{k-1}y(0)-\dots-y^{(k-1)}(0)]\\&=\sum_{k=0}^{n}a_ks^kY(s)-\sum_{k=0}^{n}a_k[s^{k-1}y(0)+\dots+y^{(k-1)}(0)]\\&=P(s)\cdot Y(s)-Q(s)\\&=G(s)\end{aligned}$$

where $Y(s)=\mathcal{L}\{y\}$, $G(s)=\mathcal{L}\{g(t)\}$.

Hence we may solve for $Y(s)$:

$$Y(s)=\frac{G(s)}{P(s)}+\frac{Q(s)}{P(s)}.$$

So y may be found by

$$y = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{G(s)}{P(s)} + \frac{Q(s)}{P(s)}\right\}$$

Example

Solve $y' + 3y = 13\sin 2t$, $y(0) = 6$

Apply $\mathcal{L}$:

$$\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = 13\mathcal{L}\{\sin 2t\}$$

$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 6$$

$$\mathcal{L}\{y\} = Y(s)$$

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$$

$$\Rightarrow \quad sY(s) - 6 + 3Y(s) = \frac{36}{s^2+4}$$

Solve for $Y(s)$: after algebra,

$$Y(s) = \frac{8}{s+3} - \frac{2s}{s^2+4} + \frac{6}{s^2+4}$$

Apply $\mathcal{L}^{-1}$ to get y:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 8\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= 8e^{-3t} - 2\cos 2t + 3\sin 2t.$$

Thus the solution to the IVP is

$$y(t) = 8e^{-3t} - 2\cos 2t + 3\sin 2t.$$

Example

Solve $y'' - 3y' + 2y = e^{-4t}$, $y(0) = 1$, $y'(0) = 5$

Apply $\mathcal{L}$:

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s - 5$$

$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 5$$

$$\mathcal{L}\{y\} = Y(s)$$

$$\mathcal{L}\{e^{-4t}\} = \frac{1}{s+4}$$

Hence

$$s^2Y(s) - s - 5 - 3sY(s) - 15 + 2Y(s) = \frac{1}{s+4}$$

Solve for $Y(s)$ and use partial fractions:

$$Y(s) = -\frac{16/5}{s-1} + \frac{25/6}{s-2} + \frac{1/30}{s+4}$$

Apply $\mathcal{L}^{-1}$:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{16}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

solves the IVP.

Section 7.3 Operational Properties

- How does modifying $f(t)$ affect $\mathcal{L}\{f(t)\}$?

7.3.1 Translation in s

Theorem

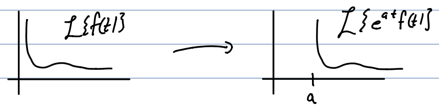
If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$.

Proof:

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{at}f(t)e^{-st}\,dt = \int_0^\infty f(t)e^{-t(s-a)} = F(s-a).$$

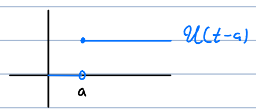
Corollary

$F(s) = \mathcal{L}\{f(t)\} \Rightarrow \mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t).$



7.3.2 Translation in t

Heaviside function: $\mathcal{U}(t-a) = \begin{cases} 0 & 0 \le t < a \\ 1 & t \ge a \end{cases}$



Theorem

If $F(s) = \mathcal{L}\{f(t)\}$, $a > 0$, then

(i) $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s).$

(ii) $\mathcal{L}\{f(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$

(iii) $\mathcal{L}\{\mathcal{U}(t-a)\} = \dfrac{e^{-as}}{s}$

(iv) $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$

Proof

$dt = d\xi$
$t-a = \xi$

(i) $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = \int_0^\infty f(t-a)\mathcal{U}(t-a)e^{-st}dt = \int_a^\infty f(t-a)e^{-st}dt = \int_0^\infty f(\xi)e^{-s(\xi+a)}d\xi = e^{-sa}\int_0^\infty f(\xi)e^{-s\xi}d\xi = e^{-sa}\mathcal{L}\{f(t)\}.$

(ii) Similar proof

(iii) Use $f(t)=1$.

(iv) follows by (i).

Heaviside decomposition:

- We can take a piecewise function and write it as a single function while using $\mathcal{U}$.

If

$$f(t) = \begin{cases} g(t) & 0 \le t < a \\ h(t) & t \ge a \end{cases}$$

then:

For $0 \le t < a$:

$$f(t) = g(t) = g(t) - g(t)\overset{0}{\mathcal{U}(t-a)} + h(t)\overset{0}{\mathcal{U}(t-a)}$$

$t \ge a$:

$$f(t) = h(t) = h(t)\mathcal{U}(t-a) = \overset{0}{g(t) - g(t)} + h(t)\mathcal{U}(t-a) = g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a).$$

Examples

(a) Solve

$$y'' - 6y' + 9y = t^2 e^{3t}, \quad y(0) = 2, \quad y'(0) = 17, \quad \text{note } \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Apply $\mathcal{L}$:

$$y'' - 6y' + 9y = t^2 e^{3t} \to s^2 Y(s) - s y(0) - y'(0) - 6s Y(s) + 6 y(0) + 9 Y(s) = \mathcal{L}\{t^2 e^{3t}\} = \frac{2}{(t-3)^3}$$

Solve for Y:

$$Y(s) = \frac{2}{s-3} + \frac{11}{(s-3)^2} + \frac{2}{(s-3)^5}$$

Apply $\mathcal{L}^{-1}$:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} + \frac{2}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{(s-3)^5}\right\}$$

$$= 2e^{3t} + 11te^{3t} + \frac{2}{4!} t^4 e^{3t}.$$

(b) Compute $\mathcal{L}\{(t+1)\,\mathcal{U}(t-1)\}$

By theorem (ii), $\mathcal{L}\{(t+1)\mathcal{U}(t-1)\} = \mathcal{L}\{f(t)\,\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\} = e^{-s}\,\mathcal{L}\{t+2\} = e^{-s}\mathcal{L}\{t\} + 2\mathcal{L}\{1\} = \frac{e^{-s}}{s^2} + \frac{2}{s}.$

(c) Solve

$$y' + y = f(t), \quad f(t) = \begin{cases} 1 & 0 \le t < 2 \\ t-2 & t \ge 2 \end{cases}, \quad y(0) = 0$$

Do Heaviside decomposition: $f(t) = 1 - \mathcal{U}(t-2) + (t-2)\,\mathcal{U}(t-2)$

Apply $\mathcal{L}$: $\quad y' + y = f(t) \to sY(s) - y(0) + Y(s) = \mathcal{L}\{1\} - \mathcal{L}\{\mathcal{U}(t-2)\} + \mathcal{L}\{(t-2)\,\mathcal{U}(t-2)\} = \frac{1}{s} - \frac{e^{-2s}}{s} + \frac{e^{-2s}}{s^2}.$

Solve for $Y(s)$:

$$Y(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{e^{-2s}}{s} + \frac{e^{-2s}}{s+1} + \frac{e^{-2s}}{s^2} + \frac{e^{-2s}}{s+1} - \frac{e^{-2s}}{s}$$

$$= \frac{e^{-2s}}{s^2} + \frac{2e^{-2s}}{s+1} - \frac{1}{s+1} - \frac{2e^{-2s}}{s} + \frac{1}{s}$$

Take $\mathcal{L}^{-1}$:

$$y(t) = t\,\mathcal{U}(t-2) + 2e^{-t}\,\mathcal{U}(t-2) - e^{-t} - 2\,\mathcal{U}(t-2) + 1.$$

Section 7.4 Operational Properties II

7.4.1 Derivatives of a Transform

- Formally:

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt = \int_0^\infty \frac{d}{ds}\left(f(t)e^{-st}\right)dt = -\int_0^\infty tf(t)e^{-st}dt = -\mathcal{L}\{tf(t)\}$$

Theorem

Suppose $F(s)=\mathcal{L}\{f(t)\}$ exists and $F(s)$ has n derivatives. Then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s).$$

Example

$$\mathcal{L}\{t\sin kt\} = -\frac{d}{ds}\mathcal{L}\{\sin kt\} = -\frac{d}{ds}\left\{\frac{k}{s^2+k^2}\right\} = \frac{2ks}{(s^2+k^2)^2}$$

Example

Solve $y''+16y=\cos(4t)$, $y(0)=0$, $y'(0)=1$.

After applying $\mathcal{L}$ and solving for $Y(s)$:

$$Y(s) = \frac{1}{s^2+16} + \frac{s}{(s^2+16)^2}$$

Apply $\mathcal{L}^{-1}$:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+16}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\} = \frac{1}{4}\sin 4t$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+16)^2}\right\} = \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{8s}{(s^2+16)^2}\right\} = \frac{1}{8}t\sin 4t$$ by previous example.

So $$y = \frac{1}{4}\sin 4t + \frac{1}{8}t\sin 4t.$$

7.4.1 Transformation of Integrals

- Convolution: define

$$f*g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

- In most analytical cases, this is the appropriate definition for multiplying two functions

Theorem

$\mathcal{L}\{f*g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$.

Proof

Let $\mathcal{U}(t-\tau) = \begin{cases} 0 & 0\le t<\tau \\ 1 & t\ge\tau. \end{cases}$

Note for fixed $t\ge 0$: $\int_0^\infty h(\tau)\mathcal{U}(t-\tau)d\tau = \int_0^t h(\tau)\mathcal{U}(t-\tau)d\tau + \underbrace{\int_t^\infty h(\tau)\mathcal{U}(t-\tau)d\tau}_{0}$ since here $\tau>t$.

$$\begin{aligned}
F(s)G(s) &= \left(\int_0^\infty f(\tau)e^{-s\tau}d\tau\right)\left(\int_0^\infty g(\beta)e^{-s\beta}d\beta\right) \\
&= \int_0^\infty\int_0^\infty f(\tau)g(\beta)e^{-s(\tau+\beta)}d\beta\, d\tau \qquad \tau+\beta=t \\
&= \int_0^\infty\int_\tau^\infty f(\tau)g(\tau-t)e^{-st}dt\,d\tau \\
&= \int_0^\infty\int_0^\infty f(\tau)g(\tau-t)e^{-st}\mathcal{U}(t-\tau)dt\,d\tau \\
&= \int_0^\infty\int_0^\infty f(\tau)g(\tau-t)e^{-st}\mathcal{U}(t-\tau)d\tau\,dt \\
&= \int_0^\infty\int_0^t f(\tau)g(\tau-t)e^{-st}d\tau\,dt \\
&= \int_0^\infty e^{-st}\left(\int_0^t f(\tau)g(\tau-t)d\tau\right)dt \\
&= \int_0^\infty e^{-st} f*g(t)dt \\
&= \mathcal{L}\{f*g(t)\}.
\end{aligned}$$

Example

- Let $f(t)=e^t$, $g(t)=\sin t$.
- Then $f*g(t) = \int_0^t e^t \sin(t-\tau)d\tau$
- Hence

$$\mathcal{L}\{f*g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = \frac{1}{s-1}\cdot\frac{1}{s^2+1}.$$

Transformation of an integral:

- If $g(t)=1$, then $g(\tau - t)=1$. Hence $f*g(t)=\int_0^t f(\tau)d\tau$.
- Thus

$$\mathcal{L}\{f*g(t)\} = \mathcal{L}\{f(t)\}\cdot\mathcal{L}\{g(t)\} \rightarrow \mathcal{L}\{\int_0^t f(\tau)d\tau\} = F(s)\cdot\frac{1}{s}.$$

- Conversely, $\mathcal{L}^{-1}\{\frac{F(s)}{s}\} = \int_0^t f(\tau)d\tau = \int_0^t \mathcal{L}^{-1}\{F(s)\}d\tau$

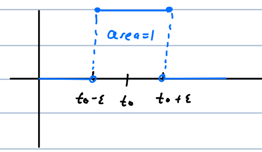
Example

$\mathcal{L}^{-1}\{\frac{1}{s(s^2+1)}\} = \mathcal{L}^{-1}\{\frac{1/(s^2+1)}{s}\} = \int_0^t \mathcal{L}^{-1}\{\frac{1}{s^2+1}\}d\tau = \int_0^t \sin\tau \, d\tau = 1-\cos t$.

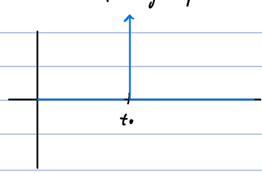
Section 7.5 The Dirac Delta Function

- Consider the box potential:

$$\delta_\varepsilon(t-t_0) = \begin{cases} 0 & 0\le t < t_0-\varepsilon \\ \frac{1}{2\varepsilon} & t_0-\varepsilon \le t \le t_0+\varepsilon \\ 0 & t > t_0+\varepsilon \end{cases}$$



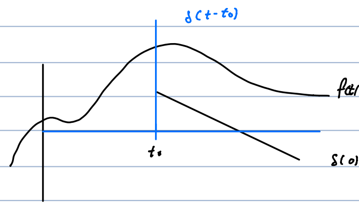
- As $\varepsilon \to 0$, the area of rectangle always equals one, but height $\to \infty$.
- In the limit, we get picture:



- Call $\delta(t-t_0) = \lim_{\varepsilon\to 0} \delta_\varepsilon(t-t_0)$.
- We think $\delta(t-t_0) = \begin{cases} \infty & t=t_0 \\ 0 & t\ne t_0 \end{cases}$
- Actually δ is not a function, despite being called the Dirac Delta function.
- It has the property: $t_0>0$, then

$$\int_0^\infty f(t)\,\delta(t-t_0)\,dt = f(t_0)$$

- Intuitively:



$\delta(0)$ has "area" $=1$, so $f(t_0)\delta(0)$ has "area" $= f(t_0)$

Theorem

For $t_0>0$, $\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty \delta(t-t_0)e^{-st}dt = e^{-st_0}$.

Non-classical Solution:

Example

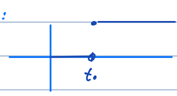
Consider the ODE

$$y' = \delta(t-t_0),\ y(0)=0$$

- The right hand side is not a function, but we can use $\mathcal{L}$ to make sense of this equation:

Applying $\mathcal{L}$: $y' = \delta(t-t_0) \rightarrow sY(s)-y(0) = \mathcal{L}\{\delta(t-t_0)\} = e^{-st_0} \rightarrow Y(s) = \frac{e^{-st_0}}{s} \rightarrow y = \mathcal{L}^{-1}\{\frac{e^{-st_0}}{s}\} = \mathcal{U}(t-t_0)$.

- The graph of $\mathcal{U}(t-t_0)$:



- Despite being a "solution" to the ODE, $\mathcal{U}(t-t_0)$ is not differentiable at $t_0 = 0$.
- This gives rise to the notion of a nonclassical solution.
- Effectively: $y' = \delta(t-t_0)$ says that $y' = \begin{cases} 0 & 0 < t < t_0,\ t > t_0 \\ +\infty & t = t_0 \end{cases}$ which makes sense with the graph of $\mathcal{U}$ in mind.