Chapter 4: Higher Order Differential Equations.

Section 4.1: Preliminary Theory - Linear Equations.

• Goal: To study Higher order differential equations

General Solution $\longrightarrow$ Solving an IVP

Example

Consider the general solution

$$y = c_1 e^{5x} + c_2 x e^{5x}$$

to the 2nd order Linear ODE:

$$y'' - 10y' + 25y = 0.$$

Find a solution satisfying

$$y(0) = 2$$
$$y'(0) = 1$$

Solution

$y = c_1 e^{5x} + c_2 x e^{5x}$ et $y(0) = 2 \Rightarrow y(0) = c_1 e^0 + c_2 \cdot 0 e^0 = c_1 = 2$

$y' = 5c_1 e^{5x} + 5c_2 x e^{5x} + c_2 e^{5x}$ et $y'(0) = 1 \Rightarrow y'(0) = 5 \cdot c_1 + c_2 = 10 + c_2 = 1 \Rightarrow c_2 = -9$

Hence the desired solution is:

$$y = 2e^{5x} - 9xe^{5x}$$

Example

Consider the general solution

$$y = c_1 e^{-x} + c_3 x + c_2$$

to

$$y''' + y'' = 0.$$

Find a solution satisfying $y(0) = 1$, $y'(0) = 0$, $y'(1) = 0$

Solution

$y = c_1 e^{-x} + c_2 x + c_3$ et $y(0) = 1 \Rightarrow y(0) = c_1 + c_3 = 1$

$y' = -c_1 e^{-x} + c_2$ et $y'(0) = 0 \Rightarrow y'(0) = -c_1 + c_2 = 0$

et $y'(\ln \frac{1}{2}) = 0 \Rightarrow y'(\ln \frac{1}{2}) = -2c_1 + c_2 = 0.$

Solving system:

$$\begin{aligned} c_1 + c_3 &= 1 \\ -c_1 + c_2 &= 0 \\ -2c_1 + c_2 &= 0 \end{aligned} \Rightarrow c_1 = 0, c_2 = 0, c_3 = 1.$$

4.1.1 Initial Value Problem and Boundary Value Problems

nth Order IVP:

(*) Solve: $\sum_{k=0}^{n} a_k(x) y^{(k)} = g(x)$ $\longrightarrow$ will have n param. family of sol.

Subject to: $y(x_0) = y_0, \dots, y^{(n-1)}(x_0) = y_{n-1}$ $\longrightarrow$ need n conditions to specify n param.

Theorem (Existence and Uniqueness)

If $a_0, \dots, a_n, g$ are continuous on interval I with $a_n(x) \neq 0$ on I, and if $x_0 \in I$ is any pt, then the IVP (*) has a unique solution on I.

Example

Find the largest interval I containing $x = 1$ for which

$$y'' + x^{-1} y = \cos(x), \quad y(1) = 1, \; y'(1) = 1$$

has a unique solution.

Solution

- Use Existence and uniqueness theorem.
  - Here $n=2$, $a_2(x)=1$, $a_1(x)=0$, $a_0(x)=x^{-1}$, $g(x)=\cos(x)$, $x_0=1$.
- Q: what is the largest interval I so that
  - (i) a_0, a_1, a_2, g are continuous on I
  - (ii) $a_2 \neq 0$ on I
  - (iii) $x_0 = 1 \in I$?
- Here a_2 is never zero, so (ii) holds.
- Only $a_0(x) = 1/x$ has a point of discontinuity at $x=0 \Rightarrow I = (-\infty, 0)$ or $(0, +\infty)$ only choices.
- But $1 \in I$ must hold $\Rightarrow I = (0, +\infty)$.

Boundary Value Problem (BVP)

Solve : $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

Subject to: $y(a) = y_0$, $y(b) = y_1$, $a \neq b$.

Called Boundary conditions (BC)

Issue: BVP may have one solution, many solutions, or no solutions.

Example

The general solution to

$$xy'' - y' = 1$$

is

$$y = c_1 x^2 + c_2 - x.$$

BC 1) $y(-1) = 1$, $y(1) = 1$

$y(-1) = 1$ at $y = c_1x^2 + c_2 - x \Rightarrow 1 = c_1 + c_2 + 1 \Rightarrow c_1 + c_2 = 0$

$y(1) = 1$ " " $\Rightarrow 1 = c_1 + c_2 - 1 \Rightarrow c_1 + c_2 = 2$

This cannot possibly hold, so no solutions can satisfy the BC 1)

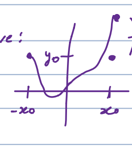
BC 2) $y(-x_0) = y_0$, $y(x_0) = y_0$, $x_0 \geq 0$

$y(-x_0) = y_0 \Rightarrow y_0 = c_1 x_0^2 + c_2 + x_0$

$y(x_0) = y_0 \Rightarrow y_0 = c_1 x_0^2 + c_2 - x_0$

Only possible if $x_0 = 0$.

Rough picture:



must avoid (x_0, y_0)

So y being a solution $\Rightarrow$ y cannot be an even function, thus restricting BC.

BC 3) $y(-1) = -1$, $y(1) = -3$

$y(-1) = -1 \Rightarrow -1 = c_1 + c_2 + 1 \Rightarrow c_1 + c_2 = -2$

$y(1) = -3 \Rightarrow -3 = c_1 + c_2 - 1 \Rightarrow c_1 + c_2 = -2$

So $c_1 = -c_2 - 2$, and $y = (-c_2 - 2)x^2 + c_2 - x$ describes all solutions satisfying this BC.

BC 4) $y(-x_0) = y_0$, $y(x_0) = y_1$, $x_0 > 0$

$y(-x_0) = y_0 \Rightarrow y_0 = c_1 x_0^2 + c_2 + x_0$

$y(x_0) = y_1 \Rightarrow y_1 = c_1 x_0^2 + c_2 - x_0$

$\Rightarrow y_0 - y_1 = 2x_0 \Rightarrow y_1 = 2x_0 - y_0$

Hence if $y(-x_0) = y_0$, then we must have $y(x_0) = 2x_0 - y_0$.

Linear Dependent functions

- functions $f_1, \dots, f_n$ are lin. dep. on interval I if there exist $c_1, \dots, c_m$ not all zero with
$$c_1 f_1(x) + \dots + c_m f_n(x) = 0 \quad \text{for all } x \in I.$$
- Idea: If this holds, then at least one f_j is a lin. comb. of the rest.

Example

$f_1(x) = x, \quad f_2(x) = e^x + x, \quad f_3(x) = 2e^x + x$

Then
$$f_1(x) - 2f_2(x) + f_3(x) = x - 2e^x - 2x + 2e^x + x = 0$$
$c_1 = 1, \; c_2 = -2, \; c_3 = 1$

So f_1, f_2, f_3 are lin. dep.

Note: $f(x) = 0$ is linearly dependent with every function:
$$c_1 f(x) + c_2 g(x) = 0$$
always holds for any c_1, and with $c_2 = 0$.

The Wronskian

- Let $f_1, \dots, f_n$ have $n-1$ derivatives. Then the Wronskian is given by
$$W(f_1, \dots, f_n)(x) = \det\begin{pmatrix} f_1(x) & \dots & f_n(x) \\ f_1'(x) & & \vdots \\ \vdots & & \\ f_1^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{pmatrix}$$

Example

Compute the Wronskian of $f_1(x) = e^x$, $f_2(x) = \cos(x)$:
$$W(e^x, \cos(x)) = \det\begin{pmatrix} e^x & \cos(x) \\ e^x & \sin(x) \end{pmatrix} = -e^x(\cos(x) + \sin(x))$$

Theorem

If $y_1, \dots, y_n$ are solutions to $a_n(x)y^{(n)} + \dots + a_0(x)y = 0$ on I, then this set is lin. indep. on I iff
$$W(y_1, \dots, y_n) \neq 0 \quad \text{for every } x \in I.$$

Remark: It is enough to check $W(y_1(x_0), \dots, y_n(x_0)) \neq 0$ for one $x_0 \in I$.

Ex

$y_1(x) = \dfrac{\cos(2\ln x)}{x^3}$, $y_2(x) = \dfrac{\sin(2\ln x)}{x^3}$ solve $x^2y'' + 7xy' + 13y = 0$ on $(0, +\infty)$.

One can show
$$W(y_1, y_2)(1) = \det\begin{pmatrix} y_1(1) & y_2(1) \\ y_1'(1) & y_2'(1) \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix} = 2 \neq 0$$

so y_1, y_2 are linearly independent.

Fundamental Set of Solutions

- $y_1, \dots, y_n$ lin. indep. solutions to $a_n(x)y^{(n)} + \dots + a_0(x)y = 0 \longleftrightarrow$ Call them a fundamental set

Theorem

- $y_1, \dots, y_n$ a fundamental set to $a_n(x)y^{(n)} + \dots + a_0(x)y = 0$ on $I \implies$ general solution is given by
$$y = C_1 y_1(x) + \dots + C_n y_n(x), \quad C_1, \dots, C_n \text{ arbitrary}.$$

Example

- $y_1 = e^{3x}$, $y_2 = e^{-3x}$ solve $y'' - 9y = 0$ on $(-\infty, \infty)$.
- We find

$$W(e^{3x}, e^{-3x}) = \det \begin{pmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{pmatrix} = -3-3 = -6 \neq 0$$

so y_1, y_2 is a fund. set.

- Hence

$$y(x) = c_1 e^{3x} + c_2 e^{-3x}$$

is the general solution.

4.1.3 Nonhomogeneous Equations

- Equations of the form

$$a_n(x) y^{(n)} + \cdots + a_0(x) y = g(x), \quad g(x) \text{ not identically zero.}$$

- Any solution y_p free of arbitrary param. is called a particular solution.

Ex

- $y'' + 9y = 27 \Rightarrow$ By inspection, $y = 3$ is a particular solution: $(3)'' + 3 \cdot 9 = 27$ ✓

Theorem

- $y_1, \dots, y_n$ fundament set to $a_n(x) y^{(n)} + \cdots + a_0(x) y = 0$
- y_p a particular sol. to $a_n(x) y^{(n)} + \cdots + a_0(x) y = g(x)$ $\quad (g(x) \not\equiv 0)$

Then

$$c_1 y_1 + \cdots + c_n y_n + y_p$$

is the gen. sol. to $a_n(x) y^{(n)} + \cdots + a_0(x) y = g(x)$

Example

Verify $c_1 e^{-x} + c_2 + x$ is the general solution to $y'' + y' = 1$.

Solution

Guess: (a) $y_1 = e^{-x}$, $y_2 = 1$ is a fund. set. to homogeneous part $y'' + y' = 0$

(b) $y_p = x$ is a particular sol. to $y'' + y' = 1$

Check: (a)

$$(e^{-x})'' + (e^{-x})' = e^{-x} - e^{-x} = 0 \quad ✓$$

$$1'' + 1' = 0 \quad ✓$$

So $e^{-x}, 1$ are sol. Must check they are lin. indep.

$$W(e^{-x}, 1) = \det \begin{pmatrix} e^{-x} & 1 \\ -e^{-x} & 0 \end{pmatrix} = e^{-x} \neq 0$$

So $e^{-x}, 1$ is a fund. set.

(b)

$$y_p'' + y_p' = 0 + 1 = 1 \quad ✓$$

So $y_p = x$ is a particular sol.

So by Theorem, gen. sol. is

$$c_1 y_1 + c_2 y_2 + y_p = c_1 e^{-x} + c_2 + x.$$

Particular solution by Inspection

- Consider homogeneous case:

$$y'' + y = 0 .$$

$y=0$ is an Obvious solution.

- Can we guess similar "easy" solutions for nonhomogeneous case?

Example

By inspection, find a particular solution:

(a) $y'' + xy = x$.

Idea: see if we can get a solution by assuming $y''=0$:

$$\cancel{y''}^{0} + xy = x \implies xy = x \implies y=1 \text{ works.}$$

(b) $y'' - 2y = 3$

Idea: same trick:

$$\cancel{y''}^{0} - 2y = 3 \implies -2y = 3 \implies y = -3/2 .$$

(c) $y'' + y' = 1$

Idea: Try $y''=0$ again:

$$\cancel{y''}^{0} + y' = 1 \implies y' = 1$$

We can solve this by inspection/integration

$$y' = 1 \implies y = x \text{ is a particular solution.}$$

Section 4.3: Homogeneous Linear Equations with Constant Coefficients

Theory:

- Consider $ay' + by = 0$, $a, b \neq 0$.

We have the Equivalence

$$ay' + by = 0 \longleftrightarrow y' + ky = 0, \quad k = \frac{b}{a} \longleftrightarrow y' = -ky$$

- The solution is $y = e^{-kx}$.
- Observe

$$y = e^{mx} \longrightarrow y^{(n)} = m^n e^{mx} = m^n y$$
$$e^{mx} \neq 0$$

- So the exponential function is something special.

- Let's generalize: consider

$$ay'' + by' + cy = 0$$

- We guess a solution $y = e^{mx}$:

$$ay'' + by' + cy = 0 \longleftrightarrow am^2 e^{mx} + bme^{mx} + ce^{mx} = e^{mx}(am^2 + bm + c) = 0 \longleftrightarrow am^2 + bm + c = 0.$$

- Therefore, if $y = e^{mx}$ is a solution, m must be a root of $am^2 + bm + c = 0$.

Example

Solve

$$y'' - 2y' - 3y = 0$$

Solution

Let $y = e^{mx}$, m to be determined.

Then $y'' - 2y' - 3y = 0$ corresponds to the equation $m^2 - 2m - 3 = (m+1)(m-3) = 0$

Thus $m = -1, 3$ and so

$$y_1 = e^{-x}, \quad y_2 = e^{3x}$$

are solutions, and since y_1, y_2 are lin. indep., the general solution is

$$y = c_1 e^{-x} + c_2 e^{3x}$$

- In full generality: consider $\sum_{k=0}^{N} a_k y^{(k)} = 0$.
- Suppose $y = e^{mx}$ is a solution. Then:

Auxiliary Equation

$$\sum_{k=0}^{N} a_k y^{(k)} = 0 \Longleftrightarrow \sum_{k=0}^{N} a_k m^k e^{mx} = 0 \Longleftrightarrow e^{mx} \sum_{k=0}^{N} a_k m^k = 0 \Longleftrightarrow \sum_{k=0}^{N} a_k m^k = 0.$$

- Therefore $y = e^{mx}$ is a solution only if m is a root of the polynomial $\sum_{k=0}^{N} a_k m^k = 0$.

- So we have the correspondence

$$\left\{ \begin{array}{l} \text{Hom. Linear ODEs} \\ \text{w/ const. real coeff.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{real} \\ \text{Polynomials} \end{array} \right\}$$

$$\left\{ \begin{array}{l} \text{solutions to} \\ \text{these ODEs} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{Solutions to} \\ \text{Polynomials} \end{array} \right\}$$

actually not a correspondence

Order 2 Case

The char. eq. is of the form: ar^2+br+c.

Three situations:

(i) char. eq. has distinct real roots

(ii) " " repeated roots

(iii) " " distinct complex roots

Case i) Let r_1, r_2 be the real distinct roots. Then the gen. sol. is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

Indeed:

$$W(e^{r_1 x}, e^{r_2 x}) = \det\begin{pmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x}\end{pmatrix} = r_2 e^{(r_1+r_2)x} - r_1 e^{(r_1+r_2)x} = (r_2 - r_1)e^{(r_1+r_2)x} \neq 0$$

So $e^{r_1 x}, e^{r_2 x}$ is a fundamental set.

Case ii) Let r be the only root. We know $y_1 = e^{r}$ is a solution. It can be shown $y_2 = xe^{r}$ is another solution. Since y_1, y_2 are lin. indep., the gen. sol. is

$$y = c_1 e^{rx} + c_2 x e^{rx}.$$

Case iii) Let $r_1 = a+bi$ and $r_2 = a-bi$, $b\neq 0$, be the roots. It can be shown that

$$y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$$

is the general solution.

Example

Solve: $y'' + 4y' + 7y = 0$

Solution: Char eq: $r^2 + 4r + 7 = 0$. Roots are $r_1 = -2+\sqrt{3}\,i$, $r_2 = -2-\sqrt{3}\,i$

So general solution is

$$y = e^{-2x}(c_1 \cos\sqrt{3}\,x + c_2 \sin\sqrt{3}\,x)$$

Higher Order

· Consider the char. eq.

$$a_n r^n + a_{n-1} r^{n-1} + \cdots a_1 r + a_0 = 0 \quad \text{for } a_n y^{(n)} + \cdots + a_0 y = 0$$

Let r be a root. We have two cases we consider:

(i) r is real and has multiplicity $k \longrightarrow e^{rx}, xe^{rx}, \ldots, x^{k-1}e^{rx}$ are lin. indep. sol.

(ii) $r = a+bi, b\neq 0$, is complex of multiplicity $1 \longrightarrow e^{ax}\cos bx, e^{ax}\sin bx$ are lin. indep. sol.

Idea: use these facts to build a fundamental set, and then the general solution.

Recall: need n lin. indep. sol. for nth order ODE.

Example

Solve

$$y^{(4)} - 4y''' + 7y'' - 6y' + 2y = 0$$

Solution

Char. Eq: $r^4 - 4r^3 + 7r^2 - 6r + 2 = 0$

Roots are $r_1 = r_2 = 1$, of mult. 2.

$r_3 = 1+i$, $r_4 = 1-i$.

Hence

$$y_1 = e^{x}, y_2 = xe^{x}, y_3 = e^{x}\cos x, y_4 = e^{x}\sin x$$

are lin. indep. sol. Since ODE has order 4, this is a fundamental set.

Hence gen. sol. is

$$y = c_1 e^{x} + c_2 x e^{x} + c_3 e^{x} \cos x + c_4 e^{x} \sin x$$

Section 4.4 Undeterminded Coefficients

- Goal: Solve nonhom. eq.:

$$a_n y^{(n)} + \cdots + a_0 y = g(x)$$

- Call the gen. sol. y_c to the hom. part

$$a_n y^{(n)} + \cdots + a_0 y = 0$$

the complementary function.

- By previous theorem, need to find y_c and a particular solution y_p to nonhom. eq.

  Recall: gen. sol. to nonhom $= y_c + y_p$.

Method of Undetermined Coefficients

Idea: based on g, do an educated "guess and check".

- We consider only the case g is a lin. comb. of:

$$P(x) = \text{polynomial}, \quad P(x)e^{\alpha x}, \quad P(x)\cos\alpha x, \quad P(x)\sin\beta x$$

- Heuristic of method:
  - If F is such a function, so is $a_n F^{(n)} + \cdots + a_0 F$.
  - Hence, if g is such a function and

$$a_n y^{(n)} + \cdots + a_0 y = g(x)$$

  we might guess y takes the same form as g.

Example

Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6$

Solution:

- By auxiliary eq., we find $y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}$ is the comp. function that solves $y'' + 4y' - 2y = 0$.
- Here $g(x) = 2x^2 - 3x + 6$. So we guess $y_p = Ax^2 + Bx + C$.
- Then

$$y_p = Ax^2 + Bx + C$$
$$y_p' = 2Ax + B \longrightarrow y_p'' + 4y_p' - 2y_p = 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 2x^2 - 3x + 6$$
$$y_p'' = 2A$$

- Polynomials uniquely defined by coeff. Hence

$$-2Ax^2 + (8A - 2B)x + 2A + 4B - 2C = 2x^2 - 3x + 6 \longrightarrow \begin{aligned} -2A &= 2 \\ 8A - 2B &= -3 \\ 2A + 4B - 2C &= 6 \end{aligned}$$

- Solving this system:

$$A = -1, \quad B = -\frac{5}{2}, \quad C = -9.$$

Hence

$$y_p = -x^2 - \frac{5}{2}x - 9$$

and so the gen. sol. is

$$y = y_c + y_p = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x} - x^2 - \frac{5}{2}x - 9$$

Example

Solve $y'' - y' + y = 2\sin 3x$

Solution:

- Find y_c first.

correction term

- We could try (and fail) guessing $y_p = A\sin 3x$. But y_p, y_p', y_p'' consisting of sines and cosines $\longrightarrow$ try $A\cos 3x + B\sin 3x$
- $g(x) = 2\sin 3x \longrightarrow$ guess $y_p = A\cos 3x + B\sin 3x$

$$y_p' = -3A\sin 3x + 3B\cos 3x$$
$$y_p'' = -9A\cos 3x - 9B\sin 3x$$

Hence
$$y_p'' - y_p' + y_p' = (\underset{\triangle}{-8A - 3B})\cos 3x + (\underset{\circ}{3A - 8B})\sin 3x = 2\sin 3x = \underset{\triangle}{0}\cdot\cos 3x + \underset{\circ}{2}\sin 3x$$

So
$$-8A - 3B = 0, \quad 3A - 8B = 2 \longrightarrow A = \frac{6}{73}, \quad B = -\frac{16}{73}$$

Hence the gen. sol. is
$$y = y_c + \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x$$

Example

$y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$

We guess $y_p = y_{p_1} + y_{p_2}$, y_{p_1} for linear term, y_{p_2} for exponential term.

$$4x - 5 \longleftrightarrow y_{p_1} = Ax + B$$
$$6xe^{2x} \longleftrightarrow y_{p_2} = Cxe^{2x} + De^{2x}$$

Correction term.

(we could try $y_{p_2} = Cxe^{2x}$, but $y_{p_2}, y_{p_2}', y_{p_2}''$ consists of xe^{2x} and e^{2x} terms, so we guess $y_{p_2} = Cxe^{2x} + De^{2x}$)

Then
$$y_p'' - 2y_p' - 3y_p = \underset{\triangle}{-3Ax} \underset{\circ}{- 2A - 3B} \underset{\square}{- 3Cxe^{2x}} + \underset{\sim}{(2C - 3E)}e^{2x} = \underset{\triangle}{4x} \underset{\circ}{- 5} + \underset{\square}{6xe^{2x}} + \underset{\sim}{0}\cdot e^{2x}$$

$$-3A = 4, \quad -2A - 3B = -5, \quad -3C = 6, \quad 2C - 3E = 0$$

$$\hookrightarrow A = -\frac{4}{3}, \quad B = \frac{23}{9}, \quad C = -2, \quad E = -\frac{4}{3}$$

So we can construct gen. sol.

$$y = y_c + y_p = y_c + y_{p_1} + y_{p_2}.$$

Example

$y'' - 5y' + 4y = 8e^x$

Here $y_c = c_1e^x + c_2e^{4x}$ contains Ae^x, so $y_p = Ae^x$ cannot work: $(Ae^x)'' - 5(Ae^x)' + 4Ae^x = 0$

When this happens, guess $y_p = Axe^x$ to correct this issue.

$$y_p'' - 5y_p' + 4y_p = -3Ae^x = 8e^x \rightarrow y_p = -\frac{8}{3}xe^x.$$

• General Guess: If $g = g_1 + \cdots + g_r$, with each g_i of different form, we guess $y_p = y_{p_1} + \cdots + y_{p_i}$ with y_{p_i} assuming the same form as g_i.
Note this guess won't always work. We modify as described later.

Case 1 no duplication | Case 2 Duplication
$y_c = c_1 e^x + c_2 e^{2x}$ | $y_c = c_1 e^x + c_2$
$y_p = e^{-x} + 1$ | $y_p = e^{-x} + 1$

Example
If $g(x) = x + \cos x + e^x = g_1 + g_2 + g_3$, we guess $y_p = Ax + B + C\cos x + De^x = y_{p_1} + y_{p_2} + y_{p_3}$.

We have two cases:
Case 1: No function in assumed y_p is a sol. of the associated hom. DE.
General Rule for Case 1: The form of y_p is a lin. comb. of all lin. indep. functions that are generated by repeated differentiation of g.

Example
$y'' + 4y = x\cos x \rightarrow g(x) = \underline{x\cos x}$, $g'(x) = \underline{\cos x} - \underline{x\sin x}$, $g''(x) = -\underline{2\sin x} - x\cos x$
So assume $y_p = Ax\cos x + B\cos x + Cx\sin x + D\sin x$.

Case 2: A function in assumed y_p is a sol. of associated hom. DE.
General Rule for Case 2: If y_{p_i} in $y_p = y_{p_1} + \cdots + y_{p_n}$ contains terms duplicating terms in y_c, then we guess $x^n y_{p_i}$ in place of y_{p_i} where n = smallest pos. integer eliminating the duplication.
guessed as in Case 1.

Example
$y'' + y = 4x + 10\sin x \rightarrow g(x) = \overset{g_1}{4x} + \overset{g_2}{10\sin x}$, $g' = 4 + 10\cos x$
We find $y_c = c_1\cos x + c_2\sin x$ and guess $y_p = Ax + B + C\cos x + D\sin x = y_{p_1} + y_{p_2}$. But terms in y_{p_2} duplicate terms in y_c so we replace y_{p_2} by $x y_{p_2}$ and take $y_p = Ax + B + Cx\cos x + Dx\sin x$

Note: These cases apply to higher order ODEs and the generalization to that setting is the natural one.

Example
$y''' + y'' = e^x\cos x$
$m^3 + m^2 \rightarrow m_1 = m_2 = 0,\ m_3 = -1 \rightarrow y_c = c_1 + c_2 x + c_3 e^{-x}$
$g(x) = \underline{e^x\cos x}$, $g'(x) = e^x\cos x - \underline{e^x\sin x} \rightarrow y_p$ is a lin. comb. of $e^x\cos x$, $e^x\sin x$
so our guess is
$y_p = Ae^x\cos x + Be^x\sin x$.
No duplication!

Further $g^{(k)}$ will consist only of $e^x\cos x$, $e^x\sin x$, so we stop here.

Example
$y^{(4)} + y''' = 1 - x^2 e^{-x}$
$y_c = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x} = y_{c_1} + y_{c_2}$
$g(x) = 1 - x^2 e^{-x} = g_1(x) + g_2(x)$, $g' = -2xe^{-x} + x^2 e^{-x}$, $g'' = -2e^{-x} + 2xe^{-x} + 2xe^{-x} - 2x^2 e^{-x}$
So guess
$y_p = A + Bx^2 e^{-x} + Cxe^{-x} + De^{-x} = y_{p_1} + y_{p_2}$

To remove duplication, take $x^3 y_{p_1}$, $x y_{p_2}$ in place of y_{p_1}, y_{p_2}.

So we use

$$y_p = x^3 c_1 + c_2 x^2 + c_3 x^3 + c_4 x e^{-x}.$$

Section 4.6 Variation of Parameters

- Another method of solving certain ODEs analytically.
- Generalizes method of integrating factors.
- Focus only on

$$a_2 y'' + a_1 y' + a_0 y = g(x)$$

Idea: Let $y_c = c_1 y_1 + c_2 y_2$ be the complementary solution.
Want to find y_p to form gen. sol. $y = y_c + y_p$.
We guess

$$y_p = u_1(x) y_1 + u_2(x) y_2$$

where we have replaced the fixed parameters c_1, c_2, with varying param. u_1, u_2.

- Since the nonhom. eq. and its associated hom. eq. depend on y, y', y'' in same way, we might expect y_c and y_p to be related in this way.

- After some math... (See text)

1st: convert $a_2 y'' + a_1 y' + a_0 y = g(x) \to y'' + Py' + Qy = f(x)$.
Then one finds u_1, u_2 satisfy

$$u_1' = -\frac{y_2 f(x)}{W}, \quad u_2' = \frac{y_1 f(x)}{W}$$

where

$$W = W(y_1, y_2) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

Thus

$$u_1 = -\int \frac{y_2 f(x)}{W}, \quad u_2 = \int \frac{y_1 f(x)}{W}$$

Then

$$y_p = u_1 y_1 + u_2 y_2 \to \text{gen. sol.}: y = y_c + y_p.$$

Example

Solve: $y'' - 4y' + 4y = (x+1)e^{2x}$

Solution:

1) Find y_c: aux. eq.: $m^2 - 4m + 4 \to y_c = c_1 e^{2x} + c_2 x e^{2x} \to y_1 = e^{2x}, y_2 = xe^{2x}$.

2) Find W:

$$W = W(y_1, y_2) = \det\begin{pmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{pmatrix} = e^{4x}$$

3) Find u_1, u_2: Here $f(x) = (x+1)e^{2x}$. So

$$u_1' = \frac{-(x+1)e^{2x} \cdot xe^{2x}}{e^{4x}} = -x^2 - x$$

$$u_2' = \frac{(x+1)e^{4x}}{e^{4x}} = x+1$$

By integration:

$$u_1 = \int u_1' = -\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2$$

$$u_2 = \int u_2' = \tfrac{1}{2}x^2 + x$$

4) Construct y_p:

$$y_p = u_1 y_1 + u_2 y_2 = \left(-\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right)e^{2x} + \left(\tfrac{1}{2}x^2 + x\right)xe^{2x}$$

5) Construct gen. sol.

$$y = y_c + y_p.$$

Example

Solve $4y'' + 36y = \csc 3x$

Solution

0) Convert: $4y'' + 36y = \csc 3x \longrightarrow y'' + 9y = \frac{1}{4}\csc 3x = f(x)$

1) $y_c = C_1 \cos 3x + C_2 \sin 3x \longrightarrow y_1 = \cos 3x,\ y_2 = \sin 3x$

2)

$W = W(y_1, y_2) = 3$

3)

$u_1' = -\frac{y_2 f(x)}{W} = -\frac{1}{12}, \quad u_2' = \frac{y_1 f(x)}{W} = \frac{1}{12}\frac{\cos 3x}{\sin 3x}$

$\hookrightarrow u_1 = \int u_1' = -\frac{1}{12}x, \quad u_2 = \int u_2' = \frac{1}{36}\ln|\sin 3x|$

4) $y_p = -\frac{1}{12}x\cos 3x + \frac{1}{36}\ln|\sin 3x|\sin 3x$

5) $y = y_c + y_p$.