8.2 Homogeneous Linear systems.

Introduction

- Let $X(t)$ be a $n\times 1$ vector of functions. Define the function D by
$$D(X(t)) = X'(t)$$
- So D takes the vector of functions $X(t)$ to its derivative $X'(t)$.

Ex

Let
$$X_1(t) = \begin{pmatrix} e^t \\ t \end{pmatrix}, \quad X_2(t) = \begin{pmatrix} 2 \\ t \\ t^2 \end{pmatrix}.$$
Then
$$D(X_1(t)) = \begin{pmatrix} e^t \\ 1 \end{pmatrix} = X_1'(t) \quad D(X_2(t)) = \begin{pmatrix} 0 \\ 1 \\ 2t \end{pmatrix} = X_2'(t).$$

- Q: Does D have eigenvalues/eigenvectors? Can we solve
$$X'(t) = D(X(t)) = \lambda X(t) ?$$
- Focus on
$$X'(t) = \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = \begin{pmatrix} \lambda x_1(t) \\ \vdots \\ \lambda x_n(t) \end{pmatrix} = \lambda X(t).$$
- Equivalently
$$\begin{matrix} x_1'(t) = \lambda x_1(t) \\ \vdots \\ x_n'(t) = \lambda x_n(t) \end{matrix} \longrightarrow \begin{matrix} x_1(t) = k_1 e^{\lambda t} \\ \vdots \\ x_n(t) = k_n e^{\lambda t} \end{matrix} \qquad (\text{Recall } y'(t) = \lambda y(t) \rightarrow y = ke^{\lambda t})$$
- So
$$X(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = \underset{n\times 1}{K} e^{\lambda t}$$
- Want to study
$$D(X(t)) = X'(t) = AX(t), \quad A \text{ a constant matrix.}$$
- If $D(X(t)) = \lambda X(t)$, then $AX(t) = \lambda X(t) \Longleftrightarrow AKe^{\lambda t} = \lambda K e^{\lambda t} \Longleftrightarrow AK = \lambda K$
- So λ is an eigenvalue of A, and K an eigenvector.
- Therefore, solutions to
$$X'(t) = AX(t)$$
may be found using the eigenvalues and eigenvectors of A.

Distinct Real Eigenvalues

Theorem

- Consider the system $X'(t) = AX(t)$, where A is a constant matrix.
- Let $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ be n distinct real eigenvalues of A, and let $K_1, \dots, K_n$ be the corresponding eigenvectors.
- Then the general solution to $X'(t) = AX(t)$ on $I = (-\infty, \infty) = \mathbb{R}$ is
$$X(t) = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t} + \dots + c_n K_n e^{\lambda_n t}$$
for any constants $c_1, \dots, c_n$.

Ex

Find the general solution to

$$x'(t) = 2x(t) + 3y(t)$$
$$y'(t) = 2x(t) + y(t)$$

Solution:

- The corresponding matrix equation is

$$X'(t) = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} X(t) = A X(t).$$

- We find the eigenvalues and eigenvectors of A, and the apply the theorem:

$$\det\begin{pmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix} = (2-\lambda)(1-\lambda) - 6 = 2 + \lambda^2 - \lambda - 2\lambda - 6 = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1)$$

So the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 4$.

Finding eigenvectors:

$\underline{\lambda_1 = -1}$

$$\left(\begin{array}{cc|c} 2-(-1) & 3 & 0 \\ 2 & 1-(-1) & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 3 & 3 & 0 \\ 2 & 2 & 0 \end{array}\right) \longrightarrow \begin{pmatrix} 3 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow 3k_1 + 3k_2 = 0 \longrightarrow k_1 = -k_2$$

So

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} -k_2 \\ k_2 \end{pmatrix} = k_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{describes all eigenvectors for } \lambda_1 = -1.$$

We choose $k_2 = 1$ so $K_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is our choice eigenvector, and $X_1(t) = K_1 e^{\lambda_1 t} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$

$\underline{\lambda_2 = 4}$

$$\left(\begin{array}{cc|c} 2-4 & 3 & 0 \\ 2 & 1-4 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} -2 & 3 & 0 \\ 2 & -3 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} -2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \to -2k_1 + 3k_2 = 0 \to k_1 = \tfrac{3}{2} k_2$$

So

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2}k_2 \\ k_2 \end{pmatrix} = k_2 \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad \text{describes all eigenvectors for } \lambda_2 = 4.$$

We choose $k_2 = 2$ so $K_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is our choice eigenvector and $X_2(t) = K_2 e^{\lambda_2 t} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$

- By the theorem, the general solution is

$$X(t) = c_1 X_1(t) + c_2 X_2(t)$$

$$= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}.$$

Ex

Solve

$$X'(t) = \begin{pmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{pmatrix} X(t).$$

Solution

Eigenvalues:

$$\begin{matrix} -4-\lambda & 1 & 1 & -4-\lambda & 1 \\ 1 & 5-\lambda & -1 & 1 & 5-\lambda \\ 0 & 1 & -3-\lambda & 0 & 1 \end{matrix}$$

$$\det\begin{pmatrix} -4-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 0 & 1 & -3-\lambda \end{pmatrix} = (-4-\lambda)(5-\lambda)(-3-\lambda) + 0 + 1 - 0 - (1)(-1)(-4-\lambda) - (-3-\lambda)(1)(1)$$
$$= -(\lambda+3)(\lambda+4)(\lambda-5) = 0$$

and so the eigenvalues are
$\lambda_1 = -3$, $\lambda_2 = -4$, $\lambda_3 = 5$.

Finding eigenvectors:

$\underline{\lambda_1 = -3}$

$$\left(\begin{array}{ccc|c} -4-(-3) & 1 & 1 & 0 \\ 1 & 5-(-3) & -1 & 0 \\ 0 & 1 & -3-(-3) & 0 \end{array}\right) = \left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & 8 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So $k_1 + k_3 = 0 \longrightarrow k_1 = -k_3$.
$k_2 = 0$

Hence $\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -k_3 \\ 0 \\ k_3 \end{pmatrix} = k_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $k_3 \in \mathbb{R}$ describes all eigenvectors for $\lambda_1 = -3$.

Thus, choosing $k_3 = 1$

$$X_1 = K_1 e^{\lambda_1 t} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-3t} \text{ is a solution, so } X_1(t) = K_1 e^{\lambda_1 t} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-3t}.$$

In a similar way, we find eigenvectors

$$K_2 = \begin{pmatrix} 10 \\ -1 \\ 1 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix} \quad \text{for } \lambda_2 = -4, \ \lambda_3 = 5.$$

Hence

$$X_2(t) = K_2 e^{\lambda_2 t} = \begin{pmatrix} 10 \\ -1 \\ 1 \end{pmatrix} e^{-4t}, \quad X_3(t) = K_3 e^{\lambda_3 t} = \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix} e^{5t}$$

are solutions.

Therefore the general solution is

$$X(t) = c_1 X_1(t) + c_2 X_2(t) + c_3 X_3(t)$$
$$= c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 10 \\ -1 \\ 1 \end{pmatrix} e^{-4t} + c_3 \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix} e^{5t}.$$

Repeated Eigenvalues

- If A is a $n \times n$ matrix, A might not have n distinct eigenvalues.
- Let's see how this can cause trouble.

Example

- Consider the system

$$X'(t) = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix} X(t) = A X(t)$$

- We find $\det(A - \lambda I) = (\lambda+3)^2 = (\lambda+3)(\lambda+3)$, and so $\lambda_1 = \lambda_2 = -3$ is the only eigenvalue.
- For this eigenvalue we only get eigenvectors

$$K_1 = c\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad c \text{ any real nonzero number.}$$

- Hence the corresponding solutions are

$$X_1(t) = c\begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t}.$$

- But since A is 2×2, we want 2 linearly independent solutions to form a general solution.
- Therefore more work is needed.

Multiplicities

- Let A be an $n \times n$ matrix.
- Suppose

$$\det(A - \lambda I) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}$$

where $\lambda_1, \dots, \lambda_k$ are distinct, and $m_1, \dots, m_k$ are integers > 0.

- Then we say λ_i is an eigenvalue with multiplicity m_i.

Repeated Eigenvalues

- Consider

$$\frac{dY}{dt} = AY = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} Y$$

Eigenvalues:

$\det(A-\lambda I) = (\lambda+2)^2 = (\lambda+2)(\lambda+2) \Rightarrow$ Eigenvalue is $\lambda=-2$ and it is repeated

Eigenvector: Can show

$Y_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector and so

$Y_1(t) = e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a straight line solution

Goal: Want general solution, so we need two linearly independent solutions

Finding General Solution

$$\text{System} \longleftrightarrow \begin{array}{l} x'(t) = -2x+y \\ y'(t) = -2y \end{array} \longleftrightarrow y(t) = y_0 e^{-2t} \longleftrightarrow x'(t) = -2x + y_0 e^{-2t} \longleftrightarrow x(t) = y_0 t e^{-2t} + x_0 e^{-2t}$$

The Problem

Suppose A is 2×2 w/ one eigenvalue λ with multiplicity 2.

Case a) λ has two linearly indep. eigenvectors K_1, K_2.

- Then $X_1(t) = K_1 e^{\lambda t}$, $X_2(t) = K_2 e^{\lambda t}$ are linearly independent solutions, from which we get the general solution $X(t) = c_1 K_1 e^{\lambda t} + c_2 K_2 e^{\lambda t}$

Case b) We can find at most one eigenvector K_1 up to scaling.

- Then we may find at most one solution $X(t) = K_1 e^{\lambda t}$ up to scaling using previous work.
- We need two linearly independent solutions to make the general solution.
- Requires new method:

General Solution for Case b)

- Guess $X(t) = V_0 e^{\lambda t} + V_1 t e^{\lambda t}$. Note $X(0) = V_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is initial condition.
- Plug into $X'(t) = AX(t)$:

$$X'(t) = \lambda V_0 e^{\lambda t} + V_1 e^{\lambda t} + \lambda V_1 t e^{\lambda t} \rightarrow \lambda V_0 e^{\lambda t} + V_1 e^{\lambda t} + \lambda V_1 t e^{\lambda t} = AV_0 e^{\lambda t} + AV_1 t e^{\lambda t}$$
$$AX(t) = AV_0 e^{\lambda t} + AV_1 t e^{\lambda t}$$

- Comparing like terms: $\lambda V_0 + V_1 = AV_0 \rightarrow V_1 = (A-\lambda I)V_0$

  $\lambda V_1 = AV_1$ $\quad$ V_1 an eigenvector if $V_1 \neq 0$.

Thm

- A an 2×2 with repeated eigenvalue λ and one eigenvector up to scaling.

Then the general solution is

$$X(t) = V_0 e^{\lambda t} + V_1 t e^{\lambda t}$$

where

$V_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is the initial value

$V_1 = (A-\lambda I)V_0$

Remark: If $V_1 = 0$, then V_0 an eigenvector

If $V_1 \neq 0$, the V_1 an eigenvector

Example

- $\frac{dY}{dt} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} Y$, $\lambda = -2$, eigenvector is only $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ up to scaling
- Let
$$V_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$ be init cond.
- Then
$$V_1 = (A - \lambda I) V_0 = \begin{pmatrix} y_0 \\ 0 \end{pmatrix}$$
- Hence general solution is
$$Y(t) = e^{-2t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t e^{-2t} \begin{pmatrix} y_0 \\ 0 \end{pmatrix}$$

Complex Eigenvalues

- Let A be 2×2. A might have complex eigenvalues $\lambda_1 = a+bi$, $\lambda_2 = a-bi = \overline{\lambda_1}$ with $b\neq 0$.
- Eigenvectors may be complex.

Theorem (Complex Solutions)

Let A be a real $n\times n$ matrix with complex eigenvalue $\lambda_1 = a+bi$ $(b\neq 0)$.

Let K_1 be a corresponding eigenvector.

Then

$$X_1(t) = K_1 e^{\lambda_1 t}, \quad X_2(t) = \overline{K_1} e^{\overline{\lambda_1} t}$$

are solutions to $X'(t) = AX(t)$.

Theorem (Real Solutions)

Assume same setting as before.

Then

$$X_1(t) = [Re(K_1)\cos bt - Im(K_1)\sin bt]e^{at}$$
$$X_2(t) = [Im(K_1)\cos bt + Re(K_1)\sin bt]e^{at}$$

are linearly independent solutions.

Corollary

If A is 2×2, and has complex eigenvalue $\lambda_1 = a+bi$, $b\neq 0$

$$X(t) = c_1 X_1(t) + c_2 X_2(t)$$

is the general solution.

Example

Solve the IVP

$X'(t) = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} X(t)$, $X(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Eigenvalues: $\lambda_1 = 2i$, $\lambda_2 = \overline{\lambda_1} = -2i$ so $a=0$, $b=-2$.

Eigenvector: $K_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + i\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Complex solutions:

$$X_1(t) = K_1 e^{\lambda_1 t} = \left[\begin{pmatrix} 2 \\ -1 \end{pmatrix} + i\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right] e^{2it}$$
$$X_2(t) = \overline{K_1} e^{\overline{\lambda_1} t} = \left[\begin{pmatrix} 2 \\ -1 \end{pmatrix} - i\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right] e^{-2it}$$

Real Solutions:

Note $Re(K_1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $Im(K_1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $a=0$, $b=-2$.

So

$$X_1(t) = \left[\begin{pmatrix} 2 \\ -1 \end{pmatrix}\cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix}\sin 2t\right]e^0$$
$$X_2(t) = \left[\begin{pmatrix} 2 \\ 0 \end{pmatrix}\cos 2t + \begin{pmatrix} 2 \\ -1 \end{pmatrix}\sin 2t\right]e^0$$

General Real solution:

$$X(t) = c_1 X_1(t) + c_2 X_2(t)$$
$$= c_1\left[\begin{pmatrix} 2 \\ -1 \end{pmatrix}\cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix}\sin 2t\right] + c_2\left[\begin{pmatrix} 2 \\ 0 \end{pmatrix}\cos 2t + \begin{pmatrix} 2 \\ -1 \end{pmatrix}\sin 2t\right]$$

Initial Value Problem:

$$X(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = c_1\begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} 2 = 2c_1 + 2c_2 \\ -1 = -c_1 \end{matrix} \rightarrow c_1 = 1,\ c_2 = 0$$

$$X(t) = \begin{pmatrix} 2\cos 2t + 2\sin t \\ -\cos 2t \end{pmatrix}$$

The Phase Plane

- $X'(t) = A\,X(t)$, A 2×2.
- A solution $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ represents a parameterized curve.
- Let $X(t) = c_1 X_1(t) + c_2 X(t)$ be the general solution. To each pair c_1, c_2 corresponds a phase curve.
- Phase Portrait : representative set of phase curves.
- Assume $\det A \neq 0$.
- Equilibrium point : The solution with $X' = 0 \Rightarrow$ solution with $AX = 0 \Rightarrow X(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, the origin.

Classifying Eq pts

1) A has distinct eigenvalues λ_1, λ_2
   - Gen. Sol. : $X(t) = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t}$

a) $\lambda_1, \lambda_2 > 0$: $X(+\infty) = c_1 K_1(+\infty) + c_2 K_2(+\infty)$, $X(-\infty) = 0$ — unstable node

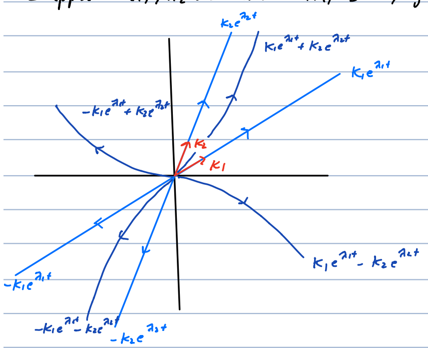
b) $\lambda_1, \lambda_2 < 0$: $X(+\infty) = 0$, $X(-\infty) = c_1 K_1(+\infty) + c_2 K_2(+\infty)$ — stable node

Note

$$X_1(t) = c_1 K_1 e^{\lambda_1 t}, \quad X_2(t) = c_2 K_2 e^{\lambda_2 t}$$

describe straight lines.

Suppose $\lambda_1, \lambda_2 > 0$. (For $\lambda_1, \lambda_2 < 0$, just reverse the arrows.) Then we may plot a couple of solution curves:



(Note $K_1 e^{\lambda_1 t}$ and $-K_1 e^{\lambda_1 t}$ are two different solution curves emanating from $(0,0)$)

c) $\lambda_1 > 0$, $\lambda_2 < 0$: Always called an unstable saddle.

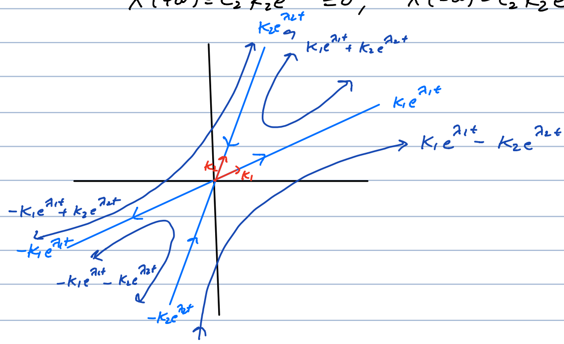
$$X(+\infty) = C_1 K_1 e^{+\infty} + C_2 K_2 e^{-\infty} = C_1 K_1(+\infty)$$
$$X(-\infty) = C_1 K_1 e^{-\infty} + C_2 K_2 e^{+\infty} = C_2 K_2(+\infty)$$

if $C_1 \neq 0$, $C_2 = 0$

$$X(+\infty) = C_1 K_1 e^{+\infty} = C_1 K_1(+\infty), \quad X(-\infty) = C_1 K_1 e^{-\infty} = 0$$

if $C_1 = 0$, $C_2 \neq 0$

$$X(+\infty) = C_2 K_2 e^{-\infty} = 0, \quad X(-\infty) = C_2 K_2 e^{+\infty} = (C_2 K_2)(+\infty)$$



1) Distinct real eigenvalues

- Evals: λ_1, λ_2, Evec: K_1, K_2
- General solution: $X(t) = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t}$
- Qualitatively plot solution curves:

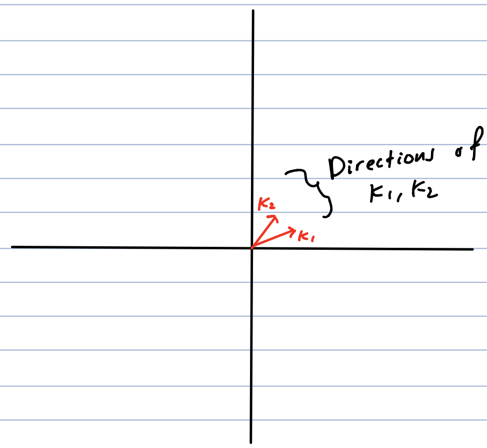
a) $\lambda_1 > 0$, $\lambda_2 > 0$:

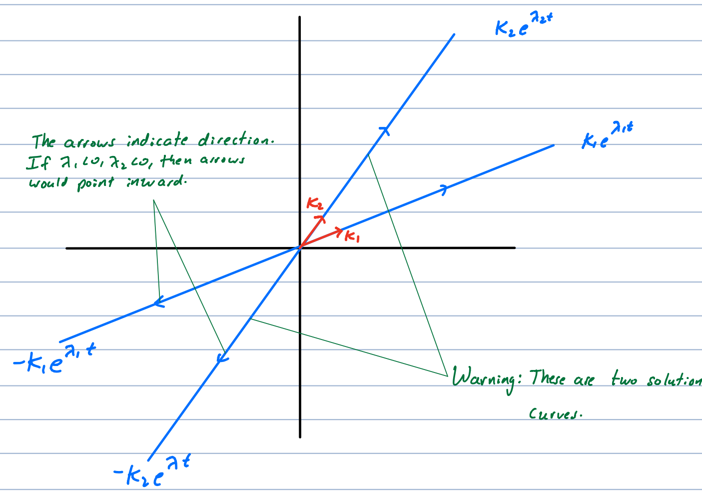
$X(+\infty) = c_1 K_1 e^{+\infty} + c_2 K_2 e^{+\infty} = (c_1 K_1 + c_2 K_2)(+\infty) \Rightarrow$ $X(t)$ asymptotically approaches the $c_1K_1 + c_2K_2$ direction "at infinity" as $t \to +\infty$

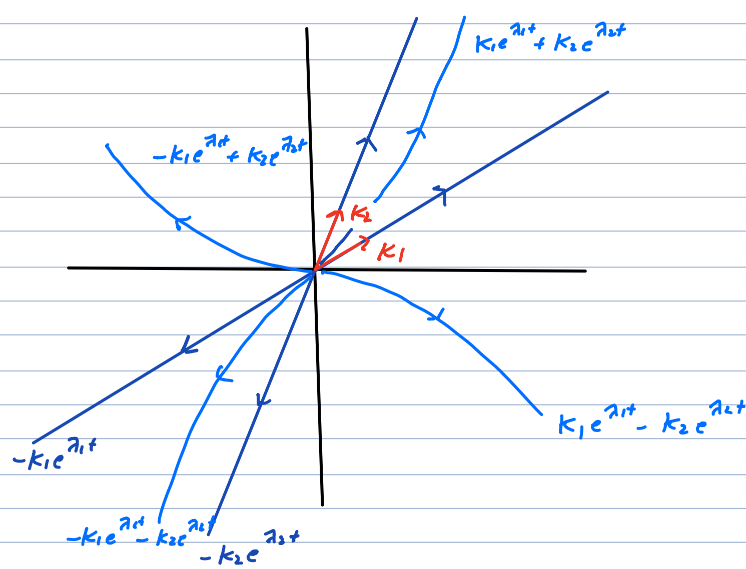
$X(-\infty) = c_1 K_1 e^{-\infty} + c_2 K_2 e^{-\infty} = (c_1 K_1 + c_2 K_2)(0) \Rightarrow$ $X(t)$ asymptotically approaches the $c_1K_1 + c_2K_2$ direction at the origin as $t \to -\infty$

$c_1 = 0$: $X(t) = c_2 K_2 e^{\lambda_2 t}$ – straight line in direction of $c_2 K_2$

$c_2 = 0$: $X(t) = c_1 K_1 e^{\lambda_1 t}$ – straight line in direction of $c_1 K_1$







Classification of Eq. Pt.

$\lambda_1 > 0, \lambda_2 > 0 \longleftrightarrow$ unstable node

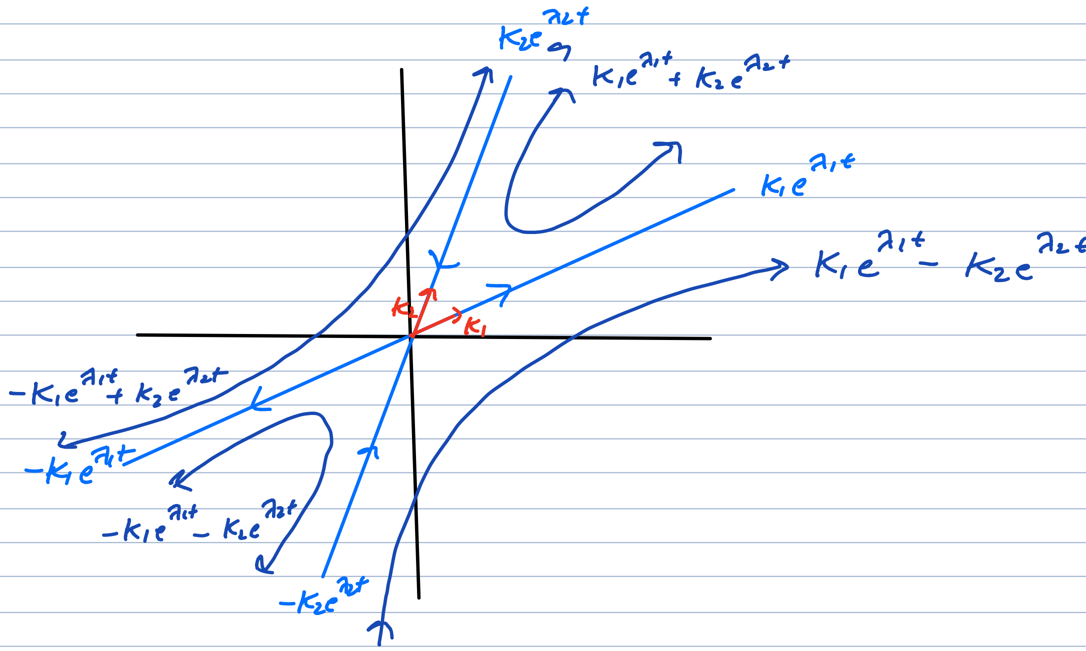
$\lambda_1 < 0, \lambda_2 < 0 \longleftrightarrow$ stable node

b) $\lambda_1 > 0$, $\lambda_2 < 0$:

$X(+\infty) = c_1 K_1 e^{+\infty} + c_2 K_2 e^{-\infty} = (c_1 K_1)(+\infty) \Rightarrow$ X(t) asymptotically approaches the $c_1 K_1$ direction "at infinity" as $t \to +\infty$

$X(-\infty) = c_1 K_1 e^{-\infty} + c_2 K_2 e^{+\infty} = (c_2 K_2)(+\infty) \Rightarrow$ X(t) asymptotically approaches the $c_2 K_2$ direction "at infinity" as $t \to -\infty$

We have same straight line solutions.



Classification of Eq. Pt.

$\lambda_1 > 0$, $\lambda_2 < 0$ $\longleftrightarrow$ unstable saddle

2) Repeated Eigenvalues:

Eval: λ (Suppose $\lambda > 0$)

a) λ has one evec up to scaling: K

General solution: $X(t) = V_0 e^{\lambda t} + V_1 t e^{\lambda t} = e^{\lambda t}(V_0 + V_1 t)$

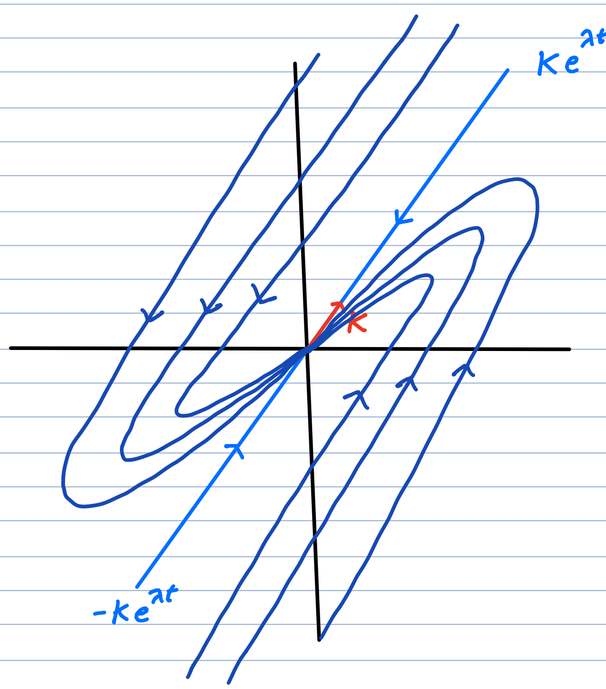
$V_1 = (A - \lambda I) V_0$

$V_1 \neq 0 \rightarrow V_1 = cK$ an evec., $V_1 = 0 \leftrightarrow V_0 = cK$ an evec.

$X(+\infty) = e^{-\infty}(V_0 + V_1(+\infty)) \approx e^{-\infty}(V_1(+\infty)) \approx 0 \cdot V_1 \rightarrow X(t)$ asymptotically approaches the origin in the V_1 direction.

$X(-\infty) = e^{+\infty}(V_0 + V(-\infty)) \approx V_1(-\infty) \rightarrow X(t)$ asymptotically approaches the $-V_1$ direction "at infinity" as $t \to -\infty$

If $V_1 = 0$, then $X(t) = V_0 e^{\lambda t}$ – straight line



Classification of Eq. Pt.

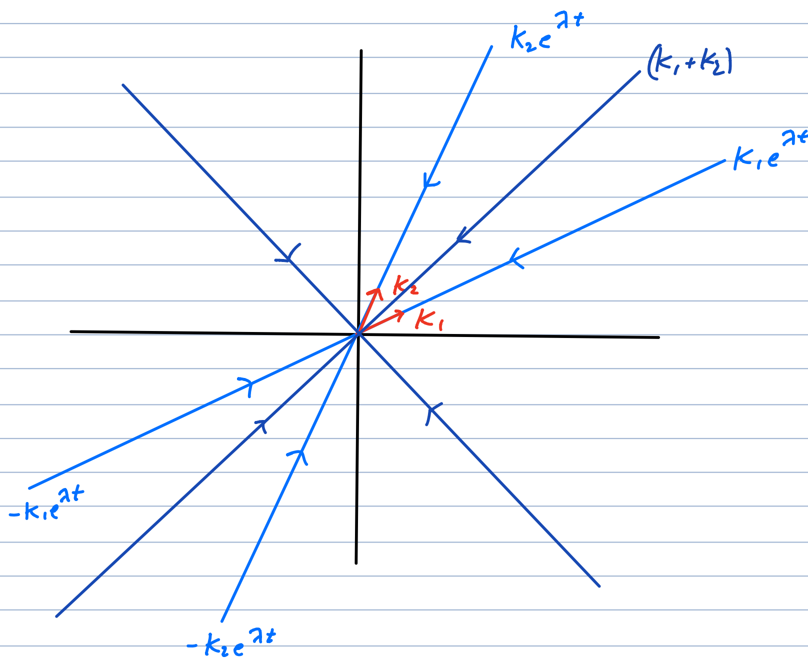
$\lambda_1 > 0 \leftrightarrow$ unstable improper node

$\lambda_1 < 0 \leftrightarrow$ stable improper node

b) λ has two linearly independendent evecs: K_1, K_2

General solution: $X(t) = c_1 K_1 e^{\lambda t} + c_2 K_2 e^{\lambda t} = (c_1 K_1 + c_2 K_2) e^{\lambda t} \overset{\text{looks like}}{=} (a e^{\lambda t}, b e^{\lambda t})$ - a straight line in direction of $c_1 K_1 + c_2 K_2$.

Suppose $\lambda < 0$



Classification of Eq. Pt.

$\lambda_1 > 0 \longleftrightarrow$ unstable proper node

$\lambda_1 < 0 \longleftrightarrow$ stable proper node

3) Complex Eigenvalues :

$\lambda_1 = a + ib$, $\lambda_2 = a - ib$, $b \neq 0$. (K an eigenvector)

a) $a = 0$

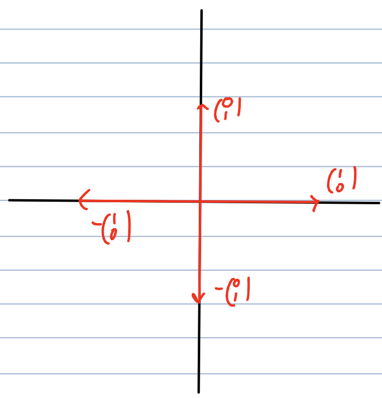
General Solution:

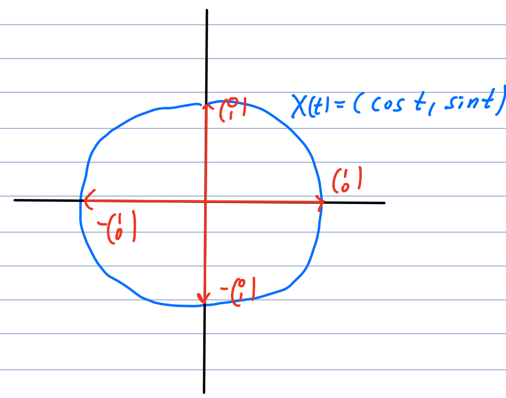
$$X(t) = (A+B)\cos bt + (A-B)\sin bt \qquad (A = c_1 \operatorname{Re} K,\ B = c_2 \operatorname{Im} K)$$

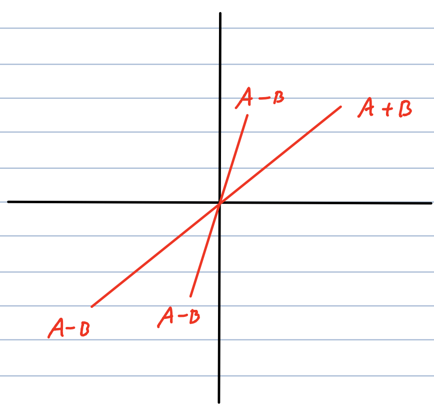
Note: $(A+B)\cos bt$ oscillates on the line connecting $A+B$ and $-A-B$

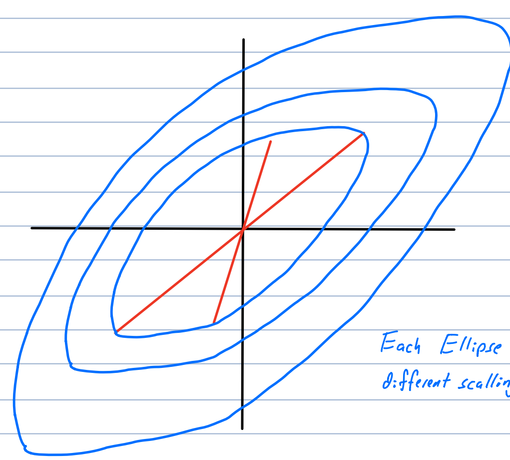
$(A-B)\sin bt$ oscillates on the line connecting $A-B$ and $-A+B$

- If $X(t) = (r\cos t, r\sin t) = \binom{1}{0} r\cos t + \binom{0}{1} r\sin t$, X traces out a circle of radius r.
- Replacing $\binom{1}{0} \to A+B$, $\binom{0}{1} \to A-B$, $X(t)$ traces an ellipse (b dictates speed and direction)









Each Ellipse corresponds to different scallings of A, B.

Classification of Eq. Pt.

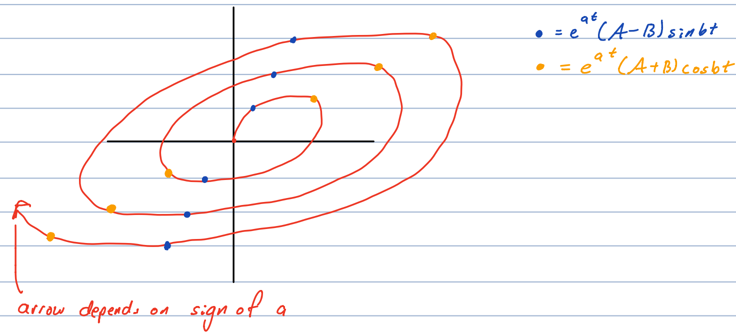
$\lambda_1 = ib \neq 0 \longleftrightarrow$ Stable center

b) $a \neq 0$

General solution : $X(t) = e^{at}\left[(A+B)\cos bt + (A-B)\sin bt\right]$

contributes changing radius

Contributes oscillation



Classification of Eq. Pt.

$\lambda_1 = a+bi,\ a>0,\ b\neq 0 \Longleftrightarrow$ unstable spiral source

$\lambda_1 = a+bi,\ a<0,\ b\neq 0 \Longleftrightarrow$ stable spiral sink.

2) A has repeated eigenvalue λ

λ has two lin. indep. eigenvectors K_1, K_2. ← (We only consider this case. If λ has only one evec up to scaling we need to do something different.)

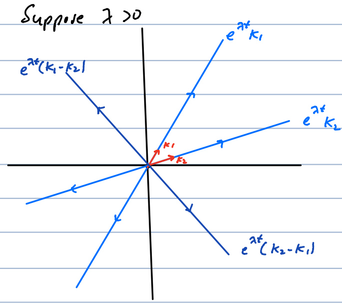
Then general solution is

$$X(t) = c_1 K_1 e^{\lambda t} + c_2 K_2 e^{\lambda t} = e^{\lambda t}(c_1 K_1 + c_2 K_2) \to \text{solution curves are straight lines.}$$

$\lambda > 0 \to$ unstable proper node

$\lambda < 0 \to$ stable proper node

Suppose $\lambda > 0$



3) Complex eigenvalues $\lambda_1 = a+ib$, $\lambda_2 = \bar{\lambda}_1$, $b \neq 0$.

General solution

$$X(t) = c_1[A\cos(bt) - B\sin(bt)]e^{at} + c_2[A\sin(bt) + B\cos(bt)]e^{at}$$

$(A = \mathrm{Re}\, K_1,\ B = \mathrm{Im}\, K_1)$

a) $\mathrm{Re}(\lambda_1) = a = 0$:

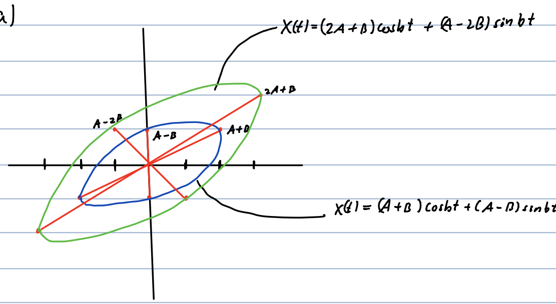
$$X(t) = (c_1 A + c_2 B)\cos(bt) + (c_2 A - c_1 B)\sin(bt) \quad \text{- stable center}$$

b) $\mathrm{Re}(\lambda_1) = a \neq 0$

$$X(t) = (c_1 A + c_2 B)\cos(bt)e^{at} + (c_2 A - c_1 B)\sin(bt)e^{at} \text{ - spiral, } a>0 \to \text{ source}$$

$a < 0 \to$ sink

a)



b) $a > 0$: The cos/sin cause oscillation, but the e^{at} ever increases the radius.