Ch 8 Systems of 1st order Linear ODEs.
8.1 Preliminary Theory

Linear systems

- Let $x_1(t), \dots, x_n(t)$ be differentiable functions, $f_1(t), \dots, f_n(t)$, $(a_{ij}(t))$ continuous all on common interval I.
- We call

$$(*) \quad \begin{aligned} \frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t) \end{aligned}$$

a linear system.

- If in $(*)$, $f_1(t) = f_2(t) = \cdots = f_n(t) = 0$, we call $(*)$ homogeneous. Otherwise it is nonhomogeneous.

Matrix Form

Ex

- Consider the system

$$x_1' = tx_1 + x_2$$
$$x_2' = 2x_1 + t^2x_2 + e^t.$$

- Convert system to matrix equation:
- Let

$$\underset{2\times1}{X(t)} = (x_i(t)) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad \underset{2\times2}{A(t)} = (a_{ij}(t)) = \begin{pmatrix} t & 1 \\ 2 & t^2 \end{pmatrix}, \quad \underset{2\times1}{F(t)} = (f_i(t)) = \begin{pmatrix} 0 \\ e^t \end{pmatrix}$$

- Then

$$X'(t) = (x_i'(t)) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}.$$

- Hence

$$X'(t) = A(t)X(t) + F(t) \iff \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} t & 1 \\ 2 & t^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \end{pmatrix} = \begin{pmatrix} tx_1 + x_2 \\ 2x_1 + t^2x_2 + e^t \end{pmatrix}$$

- In General:
- We may place $(*)$ into a matrix equation.
- Let

$$\underset{n\times1}{X(t)} = (x_i(t)), \quad \underset{n\times n}{A(t)} = (a_{ij}(t)), \quad \underset{n\times1}{F(t)} = (f_i(t))$$

- Note $\underset{n\times1}{X'} = \left(\frac{d}{dt}x_i\right)$.
- Then $(*)$ becomes

$$X'(t) = A(t)X(t) + F(t).$$

- Note that if $(*)$ is homogeneous, then $F(t) = 0$, so $(*)$ becomes

$$X'(t) = A(t)X(t).$$

Ex

Convert the matrix equation into a system:

$$X'(t) = A(t)X(t) + F(t),$$

where

$$A(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & t \\ 1 & 1 & t \end{pmatrix}, \quad F(t) = \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}, \quad X(t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Solution

$$X'(t) = A(t)X(t) + F(t) \longrightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & t \\ 1 & 1 & t \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix} = \begin{pmatrix} x + tz \\ y + tz + e^t \\ x + y + tz \end{pmatrix}$$

$$\longrightarrow \quad \begin{aligned} x' &= x + tz \\ y' &= y + tz + e^t \\ z' &= x + y + tz \end{aligned}$$

Solutions

- Consider the 1st order linear system
$$X'(t) = A(t)X(t) + F(t).$$
defined on an interval I.
- A solution to this system is a vector $X(t)$ of functions so that
$$X'(t) = A(t)X(t) + F(t)$$
holds for every $t \in I$.
- Here X is called a solution vector.

- Note that X is just a list of n scalar functions $x_1, \dots, x_n$.

Ex

Verify
$$X_1 = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}, \quad X_2 = \begin{pmatrix} 3e^{6t} \\ 5e^{6t} \end{pmatrix}$$
solve
$$X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X.$$

Solution

We find
$$X_1' = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}' = \begin{pmatrix} -2e^{-2t} \\ 2e^{-2t} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X_1 = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix} = \begin{pmatrix} -2e^{-2t} \\ 2e^{-2t} \end{pmatrix} = X_1' \quad \checkmark$$

$$X_2' = \begin{pmatrix} 3e^{6t} \\ 5e^{6t} \end{pmatrix}' = \begin{pmatrix} 18e^{6t} \\ 30e^{6t} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X_2 = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3e^{6t} \\ 5e^{6t} \end{pmatrix} = \begin{pmatrix} 18e^{6t} \\ 30e^{6t} \end{pmatrix} = X_2' \quad \checkmark$$

Note $\begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}$ has eigenvalues $-2, 6$:

$$\det\begin{pmatrix} 1-\lambda & 3 \\ 5 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - 15 = 3 - 15 + \lambda^2 - 4\lambda = \lambda^2 - 4\lambda - 12 = (\lambda+2)(\lambda-6) = 0$$
when $\lambda = -2, 6$.

Note $\begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}$ has eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix}$:

$$\lambda = -2 \qquad \left(\begin{array}{cc|c} 3 & 3 & 0 \\ 5 & 5 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow x_1 + x_2 = 0 \longrightarrow v_{-2} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Letting $x_2 = -1$ gives eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Initial-Value Problem

- Let $t_0 \in I$, and let
$$X_0 = \underset{n\times 1}{(\gamma_i)} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix}$$
be a constant vector (think of a set of constraints).
- The problem of solving
$$X'(t) = A(t)X(t) + F(t)$$
subject to
$$X(t_0) = X_0$$
is called an initial value problem.

Interpretation

- Consider the system
$$(*) \quad X'(t) = A(t)X(t) + F(t),$$
where
$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$
- A solution to this system is two functions $x = x(t)$, $y = y(t)$.
- In the xy-plane, $(x(t), y(t))$ is a parameterized curve with the velocity constrained by $(*)$
- Moreover, x and y are "coupled" by $(*)$: as x varies $\longleftrightarrow$ y varies.
- Then an initial value:
$$X(t_0) = \begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix}$$
can be understood as the "initial position" of the curve.

Superposition Principle

- Let $v_1, \dots, v_n$ be $n\times 1$ vectors, $c_1, \dots, c_n$ constants. Then
$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_2$$
is called a linear combination.

Theorem

Let $X_1, \dots, X_k$ be a set of solutions to $X'(t) = A(t)X(t)$ on I. Then the linear combination
$$X = c_1 X_1 + c_2 X_2 + \cdots + c_k X_k, \quad c_i \text{ constants},$$
is also a solution on I.

Proof

- Scaling: if X_1 is a solution to $X'(t) = A(t)X(t)$, then
$$X_1'(t) = A(t) X_1(t).$$

If $c_1 \in \mathbb{C}$, then
$$(c_1 X_1)' = c_1 X_1'.$$
So
$$(c_1 X_1)' = c_1 X_1' = c_1 (A(t) X_1(t)) = A(t)(c_1 X_1(t))$$
$\Rightarrow$ $c_1 X_1$ also solves
$$X_1 = AX.$$

◦ Summation: if X_1, X_2 are solutions, then
$$(X_1+X_2)' = X_1' + X_2'$$
and so
$$(X_1+X_2)' = X_1' + X_2' = AX_1 + AX_2 = A(X_1+X_2).$$
◦ This is enough.

Ex
Verify that if
$$X_1(t) = \begin{pmatrix} e^{x} \\ 0 \end{pmatrix}, \quad X_2(t) = \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$$
are solutions, to
$$X'(t) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} X(t)$$
then
$$2X_1(t) + 3X_2(t)$$
is also a solution.

Solution:
We compute
$$(2X_1(t) + 3X_2(t))' = 2X_1'(t) + 3X_2'(t)$$
$$= \begin{pmatrix} 2e^{t} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 6e^{2t} \end{pmatrix}$$
$$= \begin{pmatrix} 2e^{t} \\ 6e^{2t} \end{pmatrix},$$
and
$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}(2X_1(t) + 3X_2(t)) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2e^{t} \\ 3e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{t} \\ 6e^{2t} \end{pmatrix} = (2X_1(t) + 3X_2(t))' \quad \checkmark$$

Linear Dependence and Independence

◦ Vectors $v_1, v_2, \dots, v_n$ are linearly dependent if there exist constants $c_1, \dots, c_n$ not all zero such that
$$c_1v_1 + c_2v_2 + \cdots + c_nv_2 = 0.$$
Otherwise, they are called linearly independent.
◦ Idea: if $v_1, \dots, v_n$ are linearly dependent, we may solve one vector in terms of others.

Ex
Let
$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$
Then
$$-2v_1 - v_2 + v_3 = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
Hence v_1, v_2, v_3 are linearly dependent. Note $v_3 = -2v_1 - v_2$.
However
$$c_1v_1 + c_2v_2 = \begin{pmatrix} c_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
if and only if $c_1 = c_2 = 0 \Rightarrow v_1, v_2$ are linearly independent.

◦ Let $X_1, \dots, X_k$ be solutions to $X' = AX$ on an interval I. We say $X_1, \dots, X_k$ are linearly dependent on I if there are constants $c_1, \dots, c_k$, not all zero such that
$$c_1X_1 + \cdots + c_kX_k = 0.$$
Otherwise they are called linearly independent.

Wronskian

- Let $X_1, \dots, X_n$ be solutions to $X' = AX$ on an interval I.
- Define the Wronskian:

$$W(X_1(t), \dots, X_n(t)) = \det[X_1(t) \cdots X_n(t)]$$

E.g.
If $X_1(t) = \begin{pmatrix} e^t \\ 2e^t \end{pmatrix}$, $X_2(t) = \begin{pmatrix} 2e^{2t} \\ e^{2t} \end{pmatrix}$, then

$$W(X_1, X_2) = \det[X_1\ X_2] = \det\begin{pmatrix} e^t & 2e^{2t} \\ 2e^t & e^{2t} \end{pmatrix} = e^{3t} - 4e^{3t} = -3e^{3t}.$$

Theorem

The solutions $X_1, \dots, X_n$ to $X' = AX$ on I are linearly independent on I if and only if

$$W(X_1(t), \dots, X_n(t)) = \det[X_1(t) \cdots X_n(t)] \neq 0$$

- Note: it is enough to check $W(X_1(t), \dots, X_n(t)) \neq 0$ for a single $t \in I$.

Ex
Given solutions $X_1(t) = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}$, $X_2(t) = \begin{pmatrix} 3e^{6t} \\ 5e^{6t} \end{pmatrix}$ to $X'(t) = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X$, show X_1, X_2 are linearly independent using the Wronskian.

Solution:

We compute the Wronskian:

$$W(X_1(t), X_2(t)) = \det\begin{pmatrix} e^{-2t} & 3e^{6t} \\ -e^{-2t} & 5e^{6t} \end{pmatrix} = 5e^{4t} + 3e^{4t} = 8e^{4t} \neq 0 \quad \text{for all } t \in \mathbb{R}.$$

Hence X_1, X_2 are linearly independent on $I = \mathbb{R}$

Fundamental Set of Solutions

- Let $X_1, \dots, X_n$ be a linearly independent set of solution vectors to $X' = AX$ on I.

Then $X_1, \dots, X_n$ is said to be a Fundamental set of solutions.

Theorem

A fundamental set of solutions exists for the homogeneous system $X' = AX$ on some interval I.

The General Solution

Theorem

Let $X_1, \dots, X_n$ be a fundamental set of solutions to the homogeneous system $X' = AX$ on I. Then the general solution to $X' = AX$ is

$$(*) \qquad X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$$

where the c_i are arbitrary constants.

- Recall, by general solution we mean every solution to $X' = AX$ is obtainable by specifying constants $c_1, \dots, c_n$ in $(*)$.

Ex

Consider the system

$$X'(t) = \begin{pmatrix} 2t & 0 \\ t^2+2t & 2t \end{pmatrix} X(t)$$

(a) Verify

$$X_1(t) = e^{t^2}\begin{pmatrix} 1 \\ \frac{t^3}{3} \end{pmatrix} + e^{t^2}\begin{pmatrix} 0 \\ \frac{t^2}{2} \end{pmatrix}, \quad X_2(t) = \begin{pmatrix} 0 \\ e^{t^2} \end{pmatrix}$$

are linearly independent solutions.

(b) Form the general solution.

Solution:

(a) Direct computation shows they are solutions.

We compute the Wronskian

$$W(X_1(t), X_2(t)) = \det \begin{pmatrix} e^{t^2} & 0 \\ e^{t^2}\left(\frac{t^3}{3}+\frac{t^2}{2}\right) & e^{t^2} \end{pmatrix} = e^{2t^2} \neq 0.$$

Hence X_1, X_2 forms a (linearly independent) fundamental set of solutions.

(b) We take

$$X = c_1 X_1 + c_2 X_2 = \begin{pmatrix} c_1 e^{t^2} \\ c_1 e^{t^2}\left(\frac{t^3}{3}+\frac{t^2}{2}\right) \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 e^{t^2} \end{pmatrix}$$

Nonhomogeneous Systems

- Recall a nonhom. system is of the form

$$(*) \qquad X'(t) = A(t)X(t) + F(t)$$

with $F(t) \neq 0$.

- If X_p satisfies $(*)$ and is parameter free, it is called a particular solution.

Theorem

Let X_p be a particular solution to $X' = AX + F$ on I, and

$$X_c = c_1 X_1 + \cdots + c_n X_n$$

the general solution to $X' = AX$ on I. Then the general solution to $X' = AX + F$ is

$$X = X_c + X_p.$$

X_c is called the complementary function.

Ex

Verify $X_p(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^t$ is a particular solution to

$$X'(t) = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} X(t) - \begin{pmatrix} 1 \\ 7 \end{pmatrix} e^t.$$

Solution: $X_p'(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^t = \begin{pmatrix} 2e^t + te^t \\ -te^t \end{pmatrix}.$

And

$$\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^t \right] = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} te^t$$

$$= \begin{pmatrix} 3 \\ 7 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^t$$

$$= \begin{pmatrix} 3e^t + t\,e^t \\ 7e^t - te^t \end{pmatrix}.$$

Then

$$A x_p - \begin{pmatrix} 1 \\ 7 \end{pmatrix} e^t = \begin{pmatrix} 3e^t + te^t \\ 7e^t - te^t \end{pmatrix} = \begin{pmatrix} 2e^t + te^t \\ -te^t \end{pmatrix} = x_p'(t) \quad \checkmark.$$