Linear Algebra

Lecture notes

B.1 Matrices

- An $n \times m$ matrix is a list (a_{ij}) of numbers indexed by two indices i, j such that

$$1 \leq i \leq n,\quad 1 \leq j \leq m$$

- Every matrix may be placed into an array:

$$(a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

The $(a_{1j}), (a_{2j}) \cdots (a_{nj})$ are rows

$(a_{i1}), (a_{i2}) \cdots (a_{in})$ are colums.

Ex

$A = \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $\quad C = \begin{pmatrix} 1 & 3 & 4 \end{pmatrix}$ $\quad D = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 7 \end{pmatrix}$

2×2 $\quad 3 \times 1$ $\quad 1 \times 3$ $\quad 2 \times 3$

- $n \times 1$ matrices are called column matrices/vectors.

$$X = (x_i) \quad 1 \leq i \leq n,\ j = 1$$

$$= \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Operations on matrices:

Addition: given two $n \times m$ matrices $A = (a_{ij})$, $B = (b_{ij})$, define

$$A + B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}).$$

Ex

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 4 & 5 \\ 3 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = ? \begin{pmatrix} 2 & -1 \\ 0 & 2 \\ 2 & 3 \end{pmatrix}$$

Scaling: Let $c \in \mathbb{R}$, $A = (a_{ij})$. Then define

$$cA = c(a_{ij}) = (ca_{ij})$$

Ex

$c\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22}\end{pmatrix}$, $3\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2\end{pmatrix} = \begin{pmatrix} 3 & 6 & 3 \\ 3 & 6 & 6\end{pmatrix}$, $-2\begin{pmatrix} 1 & -1 \\ -1 & 1\end{pmatrix} = ? \begin{pmatrix} -2 & 2 \\ 2 & -2\end{pmatrix}$

Matrix x vector: Let $A=(a_{ij})$ be an $n\times n$ matrix, $X=(x_j)$ a $n\times 1$ vector. Then define

$$AX = \left(\sum_{j=1}^{n} a_{ij}x_j\right) = \begin{pmatrix} \sum_{j=1}^{n} a_{1j}x_j \\ \sum_{j=1}^{n} a_{2j}x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj}x_j \end{pmatrix}$$

Ex

$\begin{pmatrix} 1 & 3 \\ 2 & 1\end{pmatrix}\begin{pmatrix} 2 \\ 1\end{pmatrix} = \begin{pmatrix} 1\cdot 2+3\cdot 1 \\ 2\cdot 2+1\cdot 1\end{pmatrix} = \begin{pmatrix} 5 \\ 5\end{pmatrix}$, $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy\end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 2 & 1\end{pmatrix}\begin{pmatrix} 1 \\ 0\end{pmatrix} = ? \begin{pmatrix} 1 \\ 2\end{pmatrix}$

Matrix X Matrix: Let $A=(a_{ij})$, $B=(b_{ij})$ be $n\times n$, $n\times \ell$ matrices.
Let $\vec{b}_j = (b_{ij})$ be the jth column of B.

So $\vec{b}_1 = \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1}\end{pmatrix}$, $\vec{b}_2 = \begin{pmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2}\end{pmatrix}$, $\cdots$, $\vec{b}_\ell = \begin{pmatrix} b_{1\ell} \\ b_{2\ell} \\ \vdots \\ b_{n\ell}\end{pmatrix}$.

Then we may write

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & \cdots & b_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{n\ell}\end{pmatrix} = [\vec{b}_1 \ \vec{b}_2 \ \cdots \ \vec{b}_\ell].$$

We define

$$AB = [A\vec{b}_1 \ A\vec{b}_2 \ \cdots \ A\vec{b}_\ell]$$

Ex

$$\begin{pmatrix} 1 & 2 \\ 3 & 4\end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2\end{pmatrix} = \begin{pmatrix} 1\cdot 1+2\cdot 1 & 1\cdot 1+2\cdot 2 & 1\cdot 1+2\cdot 2 \\ 3\cdot 1+4\cdot 1 & 3\cdot 1+4\cdot 2 & 3\cdot 1+4\cdot 2\end{pmatrix}$$

$$= \begin{pmatrix} 3 & 5 & 5 \\ 7 & 11 & 11\end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix} = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22}\end{pmatrix}$$

∘ A a $m\times n$, B a $p\times q$. Then AB is only defined if $n=p$. Moreover, AB will be $m\times b$.

$$\underset{m\times n}{A}\cdot\underset{p\times q}{B} = \underset{n\times q}{AB} \text{ if } n=p.$$

<u>Matrices of functions</u>. The matrix $A(t) = (a_{ij}(t))$ where $a_{ij}(t)$ are functions is called a matrix of functions

<u>Ex</u>

$$A(t) = \begin{pmatrix} t \\ t^2+1 \\ 3 \end{pmatrix}, \quad B(t) = e^t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} e^t \\ 2e^t \\ 3e^t \end{pmatrix}, \quad C(t) = (t-1)\begin{pmatrix} t & 1 \\ 1 & t \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} t(t-1) & t-1 \\ t-1 & t(t-1) \\ 3(t-1) & 3(t-1) \end{pmatrix}$$

$$A(0) = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \quad B(0) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad C(1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

<u>Differentiating Matrices</u>: Let $A(t) = (a_{ij}(t))$, where a_{ij} are differentiable. Then define

$$\frac{dA}{dt} = \left(\frac{da_{ij}}{dt} \right).$$

<u>Ex</u>

$$A(t) = \begin{pmatrix} t & t \\ t^2 & 1 \\ t & 2 \end{pmatrix} \rightarrow \frac{dA}{dt} = \begin{pmatrix} 1 & 1 \\ 2t & 0 \\ 1 & 0 \end{pmatrix}. \qquad B(t) = \begin{pmatrix} t & t^2 \\ t^2 & t^3 \end{pmatrix} \rightarrow \frac{dB}{dt} = ? \begin{pmatrix} 1 & 2t \\ 2t & 3t^2 \end{pmatrix}$$

<u>Determinants</u>:

We only consider 2×2 and 3×3 matrices.

<u>2×2</u>: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the determinant $\det(A)$ is given by

$$\det(A) = ad - bc.$$

Graphically: Take SE diagonal of 2 entries

$$\begin{matrix} a & b \\ c & d \end{matrix} \longrightarrow ad$$

then subtract NE diagonal of 2 entries

$$\begin{matrix} a & b \\ c & d \end{matrix} \longrightarrow ad - cd = \det(A)$$

<u>Ex</u>

Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}.$$

Then

$$\det A = \det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1\cdot4 - 2\cdot3 = -2$$

$$\det B = \det \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix} = ? \quad 2\cdot8 - 4\cdot4 = 0$$

<u>3×3</u>: Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Then

$$\det(A) = a_{11} \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

How to find det(A):

① Append by first two columns:

$$\begin{array}{ccc} ① & ② & ③ \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \longrightarrow \begin{array}{ccccc} ① & ② & ③ & ① & ② \\ a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

② Add all SE diagonals of 3 entries:

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \longrightarrow a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

③ Subtract all NE diagonals of 3 entries:

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \longrightarrow \begin{aligned} &a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &- a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \end{aligned} = \det(A)$$

Ex

Let

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \longrightarrow \begin{array}{ccccc} 1 & 3 & 4 & 1 & 3 \\ 2 & 1 & 1 & 2 & 1 \\ 1 & 0 & 2 & 1 & 0 \end{array}$$

$$1\cdot1\cdot2 + 3\cdot1\cdot1 + 4\cdot2\cdot0$$
$$-1\cdot1\cdot4 - 0\cdot1\cdot1 - 2\cdot2\cdot3$$
$$= 2 + 3 + 0 - 4 - 0 - 12$$
$$= -11.$$

So $\det A = -11$.

Let

$$B = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & -1 \end{pmatrix} \longrightarrow ? \qquad \begin{array}{ccccc} 1 & 2 & 1 & 1 & 2 \\ -1 & 0 & 1 & -1 & 0 \\ 2 & 1 & -1 & 2 & 1 \end{array}$$

$$1\cdot0\cdot-1 + 2\cdot1\cdot2 + 1\cdot-1\cdot1$$
$$-2\cdot0\cdot1 - 1\cdot1\cdot1 - -1\cdot-1\cdot2$$
$$= 0 + 4 - 1 - 0 - 1 - 2$$
$$= 0$$

Let

$$C = \begin{pmatrix} 1-\lambda & 1 & -1 \\ 0 & 1-\lambda & 2 \\ 1 & 1 & 2-\lambda \end{pmatrix} \longrightarrow \begin{array}{ccccc} 1-\lambda & 1 & -1 & 1-\lambda & 1 \\ 0 & 1-\lambda & 2 & 0 & 1-\lambda \\ 1 & 1 & 2-\lambda & 1 & 1 \end{array}$$

$$(1-\lambda)^2(2-\lambda) + 2 + 0 + (1-\lambda) - 2(1-\lambda) - 0$$

So $\det C = (1-\lambda)^2(2-\lambda) + 2 - (1-\lambda)$

B.2 Gauss-Jordan Elimination

- Goal: use matrices to solve systems of linear equations.

Reduced Row Echelon form

- Elementary Row Operations on a matrix

R_{ij} - swap rows i and j : $\begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 3 & 4 \end{pmatrix} \xrightarrow{R_{13}} \begin{pmatrix} 3 & 3 & 4 \\ 2 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$

cR_i - scale row i by c : $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \xrightarrow{2R_1} \begin{pmatrix} 2 & 4 \\ 3 & 4 \end{pmatrix}$

$cR_i + R_j$ - scale row i by c the add to row j : $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 0 & 2 \end{pmatrix} \xrightarrow{-1R_3+R_1} \begin{pmatrix} 0 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 0 & 2 \end{pmatrix}$

- A matrix is in reduced row echelon form (rref) if

(i) 1st nonzero entry in nonzero row is 1. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ Bad!

(ii) The 1s make an echelon (stairs) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ Bad! $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(iii) Zero rows are on bottom. $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Bad! $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(iv) If column contains a first entry 1, then every other entry above and below is zero $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Bad! $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Puting a matrix into rref.

- We go by example.

Ex Put $A = \begin{pmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ -1 & 0 & 2 \end{pmatrix} = (a_{ij})$ into rref.

① Start with first column. Need a 1 at the a_{11} place

$$\begin{pmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ -1 & 1 & 2 \end{pmatrix} \xrightarrow{R_{13}} \begin{pmatrix} -1 & 1 & 2 \\ 2 & 4 & 2 \\ 0 & 2 & 4 \end{pmatrix} \xrightarrow{-1R_1} \begin{pmatrix} 1 & 1 & -2 \\ 2 & 4 & 2 \\ 0 & 2 & 4 \end{pmatrix}$$

Now need zeros below:

$$\begin{pmatrix} 1 & 1 & -2 \\ 2 & 4 & 2 \\ 0 & 2 & 4 \end{pmatrix} \xrightarrow{-2R_1+R_2} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 4 & 6 \\ 0 & 2 & 4 \end{pmatrix}$$

② Move to 2nd column. Need a 1 at the a_{22} place and zeros above and below.

$$\begin{pmatrix} 1 & 1 & -2 \\ 0 & 4 & 6 \\ 0 & 2 & 4 \end{pmatrix} \xrightarrow{\frac{1}{4}R_2} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 3/2 \\ 0 & 2 & 4 \end{pmatrix} \xrightarrow{-2R_2+R_3} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-R_2+R_1} \begin{pmatrix} 1 & 0 & -7/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{pmatrix}$$

③ Finish up with 3rd column.

$$\begin{pmatrix} 1 & 0 & -7/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-\frac{3}{2}R_3+R_2} \begin{pmatrix} 1 & 0 & -7/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\frac{7}{2}R_3+R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

System to Matrix Conversions

- Observe that to any system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

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$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

we may associate a matrix equation:

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}_{B}$$

/

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$A \cdot X = B$$

- We use the augmented matrices

$$(A|B) = \left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right) \qquad (A|B) = \left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right)$$

to solve the respective linear system.

Ex

(a) Convert the system

$$x - y + z = 1$$
$$x + y + 2z = 0$$
$$x + z = -1$$

to an augmented matrix.

(b) Convert the matrix

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 1 \end{pmatrix}$$

to a linear system.

(c) Convert the system

$$x + 2y = 1$$
$$y + 2x = -1$$

to an augmented matrix

(a)

$$\begin{array}{lcl} x - y + z = 1 & \to & 1 \quad -1 \quad 1 \quad 1 \\ x + y + 2z = 0 & \to & 1 \quad 1 \quad 2 \quad 0 \\ x + z = -1 & \to & 1 \quad 0 \quad 1 \quad -1 \end{array} \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & -1 \end{array}\right)$$

(b)

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 1 \end{pmatrix} \to \begin{array}{l} x - y = 1 \\ 2x + 2y = 1 \end{array}$$

(c)

$$\begin{array}{lcl} x + 2y = 1 & \to & 1 \quad 2 \quad 1 \\ y + 2x = -1 & & 2 \quad 1 \quad -1 \end{array} \to \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & -1 \end{array}\right)$$

Solving a System via Matrices

- To solve a system:

1. Convert linear system to augmented matrix.
2. Put augmented matrix into rref.
3. Convert resulting matrix back to linear system.
4. Done.

Ex

Solve the system

$$\begin{aligned} y - x &= 1 \\ y + x &= 3 \end{aligned}$$

$$\begin{aligned} x - y &= 1 \\ x + y &= 3 \end{aligned} \longleftrightarrow \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 1 & 3 \end{array}\right)$$

$$\downarrow \qquad\qquad \downarrow -R_1 + R_2$$

$$\begin{aligned} x - y &= 1 \\ 0 + 2y &= 2 \end{aligned} \longleftrightarrow \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 2 & 2 \end{array}\right)$$

$$\downarrow \qquad\qquad \downarrow \tfrac{1}{2}R_2$$

$$\begin{aligned} x - y &= 1 \\ y &= 1 \end{aligned} \longleftrightarrow \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 1 \end{array}\right)$$

$$\downarrow \qquad\qquad \downarrow R_2 + R_1$$

$$\begin{aligned} x &= 2 \\ y &= 1 \end{aligned} \longleftrightarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right)$$

Ex

Find all all $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ which satisfy

$$\begin{aligned} x - y + z &= 1 \\ -x - y + 2z &= 2 \\ 2x + 2y - 4z &= -4 \end{aligned}$$

Solution.

1.

$$\begin{aligned} x - y + z &= 1 \\ -x - y + 2z &= 2 \\ 2x + 2y - 4z &= -4 \end{aligned} \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -1 & -1 & 2 & 2 \\ 2 & 2 & -4 & -4 \end{array}\right)$$

2.

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -1 & -1 & 2 & 2 \\ 2 & 2 & -4 & -4 \end{array}\right) \xrightarrow{R_1 + R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & -2 & 3 & 3 \\ 2 & 2 & -4 & -4 \end{array}\right) \xrightarrow{-2R_1 + R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & -2 & 3 & 3 \\ 0 & 4 & -6 & -6 \end{array}\right) \xrightarrow{-\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 4 & -6 & -6 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 4 & -6 & -6 \end{array}\right) \xrightarrow{\frac{1}{4}R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 1 & -\frac{3}{2} & -\frac{3}{2} \end{array}\right) \xrightarrow{-R_2+R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

③ $\left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow$ The system $\begin{aligned} x - \tfrac{1}{2}z &= -\tfrac{1}{2} \\ y - \tfrac{3}{2}z &= -\tfrac{3}{2} \end{aligned}$ describes all points which satisfy the original linear system.

Hence $\begin{aligned} x &= \tfrac{1}{2}z - \tfrac{1}{2} \\ y &= \tfrac{3}{2}z - \tfrac{3}{2} \\ z &= z \end{aligned}$

We can thus compute $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}z - \frac{1}{2} \\ \frac{3}{2}z - \frac{3}{2} \\ z \end{pmatrix} = z\begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 0 \end{pmatrix}$$

where z can be any real number.

Three situations

- Consider the system

$$\begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \quad \ddots \quad \vdots \quad \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = b_n \end{matrix} \longleftrightarrow AX = B$$

(i) $AX=B$ has a unique solution X iff $\det A \neq 0$.

If $\det A = 0$,

(ii) $AX=B$ might have infinitely many solutions

(iii) $AX=B$ might have no solutions.

B.3 Eigenvalues/Eigenvectors

Complex numbers

- Define i by $i^2 = -1$. If $a, b \in \mathbb{R}$, then $a+ib$ is called a complex number.
- The set of complex numbers is denoted $\mathbb{C}$.
- Multiplication: if $a+ib$ and $x+iy$ are complex numbers, then

$$\begin{aligned}(a+ib)(x+iy) &= ax + a\cdot iy + ibx + ib\cdot iy \\ &= ax + iay + ibx + i^2 by \\ &= ax + i(ay+bx) - by \\ &= ax - by + i(ay+bx)\end{aligned}$$

Ex

$(1+3i)(1-3i) = 1 - 3i + 3i + 9 = 10.$

- Given $z = x+iy \in \mathbb{C}$, we define conjugation:

$$\bar{z} = x - iy.$$

- The modulus of $z = x+iy \in \mathbb{C}$

$$|z| = \sqrt{z\cdot\bar{z}} = \sqrt{x^2+y^2}$$

- Division: Let $z = a+ib$, $w = x+iy$. Then

$$\frac{w}{z} = \frac{w\bar{z}}{|z|^2} = \frac{(x+iy)(a-ib)}{a^2+b^2}$$

- If $z = a+ib \in \mathbb{C}$, then

Real part of $z = \operatorname{Re} z = a$

Imaginary part of $z = \operatorname{Im} z = b$

Identity Matrix

- The $n\times n$ matrix with 1s on diagonal and zero elsewhere is called the identity matrix, and is denoted by I.

e.g.,

$$I = \begin{pmatrix}1&0\\0&1\end{pmatrix}, \quad I = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

- Note $IA = A$ for all $n\times m$ matrices A.
- Theorem:

Consider the system

$$AX = 0$$

where A is $n\times n$, X $n\times 1$, B $n\times 1$.

This system as a unique solution iff $\det A \neq 0$.

- Corollary:

The system $AX=0$ has a nonzero solution iff $\det A = 0$.

Eigenvalues

- Let A be $n\times n$, $\lambda \in \mathbb{C}$. If there exists $n\times 1$ $X \neq 0$ so that

$$AX = \lambda X,$$

we call λ an eigenvalue. Any $X \neq 0$ with $AX = \lambda X$ is called an eigenvector associated with λ.

Ex

Let
$$A=\begin{pmatrix}1&2\\2&4\end{pmatrix}.$$

Verify $v_1=\begin{pmatrix}1\\2\end{pmatrix}$, $v_2=\begin{pmatrix}-2\\1\end{pmatrix}$ are eigenvectors with corresponding eigenvalues $\lambda_1=5$, $\lambda_2=0$

We compute

$$Av_1=\begin{pmatrix}1&2\\2&4\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}5\\10\end{pmatrix}=5\begin{pmatrix}1\\2\end{pmatrix}=\lambda_1 v_1 \implies \quad v_1 \text{ an eigenvector and } \lambda_1 \text{ an eigenvalue.}$$

$$Av_2=\begin{pmatrix}1&2\\2&4\end{pmatrix}\begin{pmatrix}-2\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}=0\begin{pmatrix}-2\\1\end{pmatrix}=\lambda_2 v_2 \implies \quad v_2 \text{ an eigenvector and } \lambda_2 \text{ an eigenvalue.}$$

<u>Finding Eigenvectors and Eigenvalues</u>

- We want to find $\lambda\in\mathbb{C}$, $X\neq 0$ so
$$AX=\lambda X.$$
- But
$$AX=\lambda X \iff AX-\lambda X=0 \iff AX-\lambda IX=0 \iff (A-\lambda I)X=0.$$

- Recall for $n\times n$ matrix B, $BX=0$ has nonzero solutions iff $\det B=0$.
- So $(A-\lambda I)X=0$ has nonzero solutions $\iff \det(A-\lambda I)=0$.
- Thus if we find all λ so that $\det(A-\lambda I)=0$, we will have found all eigenvalues.
- Note that $\det(A-\lambda I)=0$ will be a polynomial equation in λ. Hence the problem will reduce to that of solving the zeros of a polynomial. We call $\det(A-\lambda I)=0$ the characteristic equation.
- Now suppose we have found eigenvalue λ to A. Then to find X so that $AX=\lambda X$, we need to solve
$$(A-\lambda I)X=0.$$
- But to find solutions to the system $(A-\lambda I)X=0$, we need only to put the augmented matrix
$$(A-\lambda I\,|\,0)$$
into rref. The resulting solution
$$X=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}$$
will then be an eigenvector corresponding to eigenvalue λ.
- Note that if $AX=\lambda X$, then for any $c\in\mathbb{R}$, we have
$$A(cX)=c(AX)=c\lambda X=\lambda(cX).$$
So if X an eigenvector, so is cX.

Ex

Find eigenvalues and eigenvectors to $A=\begin{pmatrix}1&2\\2&4\end{pmatrix}$.

We want to find all $\lambda\in\mathbb{C}$ so that $(A-\lambda I)X=0$. Hence we solve the characteristic equation
$$\det(A-\lambda I)=0.$$

We compute

$$\det\left(\begin{pmatrix}1&2\\2&4\end{pmatrix}-\lambda\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)=\det\left(\begin{pmatrix}1&2\\2&4\end{pmatrix}-\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}\right)$$
$$=\det\begin{pmatrix}1-\lambda&2\\2&4-\lambda\end{pmatrix}$$
$$=(1-\lambda)(4-\lambda)-4$$
$$=4+\lambda^2-5\lambda-4$$
$$=\lambda^2-5\lambda$$
$$=\lambda(\lambda-5).$$

But $\lambda(\lambda-5)=0$ only when $\lambda=0$, $\lambda=5$. Hence

$$\lambda_1=0$$
$$\lambda_2=5$$

are the eigenvalues.

- Now let's compute the eigenvectors.

We place $(A-\lambda I)X=0$ into an augmented matrix:

$$\begin{pmatrix}1-\lambda&2\\2&4-\lambda\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \Longleftrightarrow \left(\begin{array}{cc|c}1-\lambda&2&0\\2&4-\lambda&0\end{array}\right).$$

We find the eigenvector to $\lambda_1=0$ first:

$$\left(\begin{array}{cc|c}1-\lambda_1&2&0\\2&4-\lambda_1&0\end{array}\right)=\left(\begin{array}{ccc}1&2&0\\2&4&0\end{array}\right)\xrightarrow{-2R_1+R_2}\left(\begin{array}{cc|c}1&2&0\\0&0&0\end{array}\right)\longrightarrow x_1+2x_2=0$$

Hence $x_1=-2x_2$, and so

$$X=\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}-2x_2\\x_2\end{pmatrix}=x_2\begin{pmatrix}-2\\1\end{pmatrix},\quad x_2\in\mathbb{R},\ x_2\neq0$$

describes all eigenvectors corresponding to $\lambda_1=0$.

Choosing $x_2=1$, we get that

$$v_1=\begin{pmatrix}-2\\1\end{pmatrix}$$

is an eigenvector.