Math 2410 | # Exam I | **Feb 15^{th}, 2018**

Name: ____________________________________

Instructions:

- Answer each question to the best of your ability. Show your work or receive no credit.
- All answers must be written clearly. Be sure to erase or cross out any work that you do not want graded. Partial credit can not be awarded unless there is legible work to assess.
- If you require extra space for any answer, you may use the back sides of the exam pages. Please indicate when you have done this so that I do not miss any of your work.

ACADEMIC INTEGRITY AGREEMENT

I certify that all work given in this examination is my own and that, to my knowledge, has not been used by anyone besides myself to their personal advantage. Further, I assert that this examination was taken in accordance with the academic integrity policies of the University of Connecticut.

Signed:____

(full name)

Questions:	1	2	3	4	5	6	7	8	**Total**
Score:									

From WebAssign

1. (5 points) Assume that k is a real valued constant. For which value(s) of k is $y = x^k$ a solution to the differential equation

$$x^2y'' + 4xy' - 4y = 0?$$

If no such value exists, explain why not.

$y = x^k \longrightarrow y' = kx^{k-1} \rightarrow y'' = k(k-1)x^{k-2}.$

Plugging in:

$$x^2y'' + 4xy' - 4y = 0 \longrightarrow x^2 k(k-1)x^{k-2} + 4x\,kx^{k-1} - 4x^k = 0$$

$$\rightarrow k(k-1)x^k + 4kx^k - 4x^k = 0$$

$$\rightarrow (k(k-1) + 4k - 4)x^k = 0$$

$$\rightarrow k^2 + 3k - 4 = 0$$

$$\rightarrow (k-1)(k+4) = 0$$

$$\rightarrow k = 1, -4.$$

Practice exam type problem

2. (5 points) Consider the IVP:

$$\begin{cases} y' = 3xy^{1/3} \\ y(0) = 0. \end{cases}$$

(a) Verify that both

$$y_1 = x^3$$
$$y_2 = 0$$

are solutions to the IVP.

(b) Why does the IVP having two distinct solutions not contradict the conclusion of the existence and uniqueness theorem?

(a) $y_1' = 3x^2$. $\quad 3xy_1^{1/3} = 3x(x^3)^{1/3} = 3x^2$ ✓

$y_1(0) = 0^3 = 0$ ✓

$y_2' = 0 \quad 3xy_2^{1/3} = 3 \cdot x \cdot 0 = 0$ ✓

$y_2(0) = 0$ ✓

(b) Here $f(x,y) = 3xy^{1/3}$, $\frac{\partial}{\partial y} f(x,y) = \frac{x}{y^{2/3}}$, which is disc. at $(0,0)$ $\to$ cannot apply existence/uniqueness theorem $\to$ no contradiction.

Simpler version of class/WebAssign problem / from text book problems/
from calc II worksheet.

(3)

3. (5 points) Suppose that a large mixing tank initially holds 300 gallons of water in which 50 pounds of salt have been dissolved. (1) Pure water is pumped into the tank at a rate (2) of 3gal/min, and when the solution is well stirred, it is then pumped out at the same rate. Construct an initial value problem whose solution is the amount of salt $A(t)$ in the tank at time t.

$$\frac{dA}{dt} = R_{in} - R_{out}.$$

(1) $\Rightarrow$ $R_{in} = 0$.

(2) $\Rightarrow$ Volume constant.

$R_{out} = $ conc. $\times$ 3 gal/min.

$$\text{Conc.} = \frac{\text{amount of salt}}{\text{volume}} = \frac{A(t)}{300}.$$

$$\Rightarrow R_{out} = 3 \cdot \frac{A(t)}{300} = \frac{A(t)}{100}$$

$$\Rightarrow \quad \frac{dA}{dt} = -\frac{A(t)}{100}.$$

(3) $\Rightarrow$ $A(0) = 50$.

Like problem from class.

4. (5 points) The following is a direction field for $y' = y\cos(x)$.

   (a) Sketch a solution curve through the indicated point $(2, 2)$.

   (b) If y is a solution to the IVP

$$\begin{cases} y' = y\cos(x) \\ y(2) = 2 \end{cases},$$

   use your sketched solution curve to estimate $y(1.5)$. Clearly indicate the point $(1.5, y(1.5))$ on your solution curve.

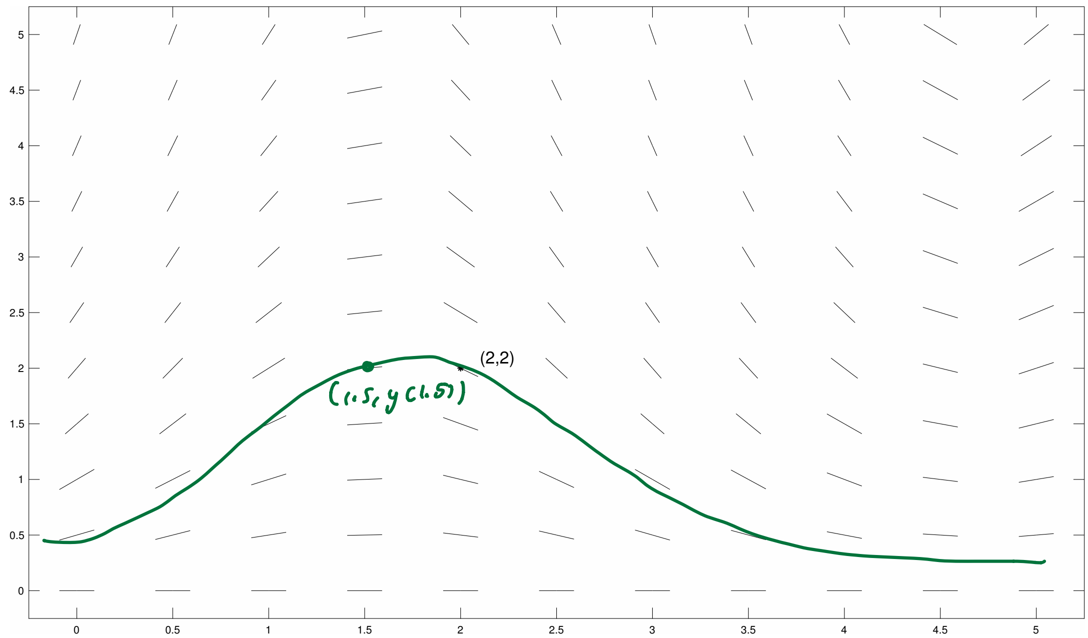

Figure 1: Direction Field

So $y(1.5) \approx 2$

General separable ODE type problem.

5. (5 points) Consider the separable differential equation

$$y' = \sin(x+y) - \sin(x-y).$$

(a) Verify that this ODE has the trivial solution.

(b) Use $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$ to solve the ODE by using a separation of variables. You may leave your answer in implicit form.
Hint: $\int \csc(x)dx = -\ln(\cot(x) + \csc(x)) + c$.

Extra credit (3 points) Can the trivial solution be obtained by specifying parameter(s) from your solution in (b)?

(a) $y=0$ is trivial solution:

$0 = 0' = \sin(x+0) - \sin(x-0) = \sin(x) - \sin(x) = 0$ ✓

(b) $y' = \sin(x+y) - \sin(x-y) = \sin(x)\cos(y) + \cos(x)\sin(y) - \sin(x)\cos(y) + \cos(x)\sin(y)$

$= 2\cos(x)\sin(y)$

$\Rightarrow \dfrac{dy}{\sin(y)} = 2\cos x\, dx \Rightarrow \int \csc(y)\, dy = \int 2\cos x\, dx$

$\Rightarrow -\ln(\cot(y) + \csc(y)) = 2\sin(x) + C$.

General linear ODE type problem

6. (5 points) Find the general solution to the linear first order equation

$$y' + \tan(x)y = \sec(x).$$

Give the solution explicitly. Hint: $\int \tan(x)dx = \ln(\sec(x)) + c$ and $\int \sec^2(x)dx = \tan(x) + c$.

Here $P(x) = \tan x$, $f(x) = \sec(x)$.

$$\mu(x) = e^{\int \tan x \, dx} = e^{\ln(\sec x)} = \sec(x).$$

$$\Rightarrow \quad (y \sec(x))' = \sec^2(x) \Rightarrow y\sec(x) = \tan(x) + C$$

$$\Rightarrow \quad y = \frac{\tan(x)}{\sec(x)} + \frac{C}{\sec(x)}$$

$$= \sin(x) + C\cos(x)$$

Example from class

7. (5 points) Use substitution to reduce the following ODE into a separation of variables, and then solve it:

$$y' = 1 + e^{y-x+5}.$$

Give the solution explicitly.

Let $u = y - x + 5 \Rightarrow u' = y' - 1 \Rightarrow y' = u' + 1.$

Substitution:

$$y' = 1 + e^{y-x+5} \Rightarrow u' + 1 = 1 + e^{u} \Rightarrow u' = e^{u}$$

$$\Rightarrow \frac{du}{e^u} = dx \Rightarrow \int e^{-u} du = \int dx \Rightarrow$$

$$-e^{-u} = x + c \Rightarrow e^{-y+x-5} = -x + c$$

$$\Rightarrow -y + x - 5 = \ln(-x + c)$$

$$\Rightarrow y = x - 5 - \ln(-x + c).$$

WebAssign problem.

8. (5 points) Consider the IVP

$$\begin{cases} y' = x + y^2 \\ y(0) = 0. \end{cases}$$

Let y be a solution to this IVP, and use Euler's method with step $h = 0.1$ to approximate $y(0.3)$. Indicate clearly all relevant $x_k, y_k, f(x_k, y_k)$. Round your final answer to 3 decimal places.

$y_n = h f(x_{n-1}, y_{n-1}) + y_{n-1}$. $\quad f(x,y) = x + y^2$

$(x_0, y_0) = (0,0)$. $\quad h = 0.1$.

$y(0.1) \approx y_1 = (0.1) f(0,0) + 0 = 0$. $\qquad x_1 = x_0 + 0.1 = 0.1$

$y(0.2) \approx y_2 = (0.1) f(0.1, 0) + 0 = (0.1)(0.1 + 0) = 10^{-2}$.

$y(0.3) \approx y_3 = (0.1) f(0.2, 10^{-2}) + 10^{-2} = (0.1)(0.2 + 10^{-2}) + 10^{-2}$

$= 0.031$