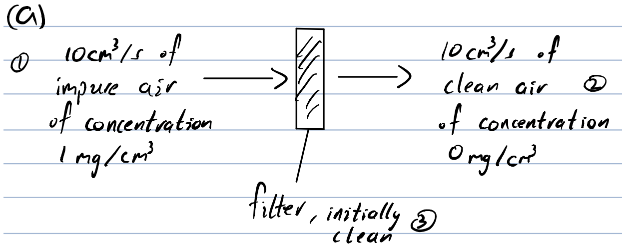
Assignment 2

(a)



④ Rate at which mass of impurities added to filter $=$ Rate of mass into filter $-$ Rate of mass out of filter

$$= \underbrace{10\,cm^3/s \cdot 1mg/cm^3}_{\text{By ①}} - \underbrace{10cm^3/s \cdot 0mg/cm^3}_{\text{By ②}}$$

$$= 10mg/s.$$

- Recall

  r.o.c. of a variable y with respect to x $= \dfrac{dy}{dx}$

- So

  Rate at which mass of impurities added to filter $= \dfrac{dI}{dt}$

⑤ • So by ④ $\dfrac{dI}{dt} = 10.$

⑥ • By ③ the initial condition is $I(0)=0$

By ⑤ and ⑥, the desired IVP is

$$\begin{cases} \dfrac{dI}{dt} = 10 \\ I(0) = 0 \end{cases}$$

- Solving the IVP:

$$dI = 10dt \Rightarrow I(t) = 10t + c$$
$$I(0)=0 \Rightarrow I(0) = 10\cdot 0 + c = 0 \Rightarrow c = 0.$$

Hence solution is

$$I(t) = 10t.$$

(b) "①Assume that the rate at which the impurities take up volume is proportional to the current volume of impurities present in the filter"

"the rate at which the impurities take up volume" => translates to $\frac{dV}{dt}$

"current volume of impurities present" => translates to $V(t)$

So ① becomes

$$\frac{dV}{dt} \text{ is proportional to } V(t).$$

This means

③ $\frac{dV}{dt} = k V(t)$ for some constant k.

④ ∘ The initial condition is explicit by ②

$$V(0) = 10^{-4} V_{filter}.$$

∘ The the IVP is (by ① and ④)

$$\begin{cases} \frac{dV}{dt} = k V(t) \\ V(0) = 10^{-4} V_{filter} \end{cases}$$

∘ Solving it

$$\frac{dV}{V} = kdt \Rightarrow \ln V = kt + c \Rightarrow V = Ae^{kt}.$$

$$V(0) = 10^{-4} V_{filter} \Rightarrow V(t) = 10^{-4} V_{filter} e^{kt}.$$

(c) As in (b), the ODE is explicitly described:

"Suppose the $\underbrace{\text{rate at which } R_{out} \text{ changes}}_{\frac{dR_{out}}{dt}}$"

"jointly and negatively proportional to two things" => there are two things, say, X and Y, so that $\frac{dR_{out}}{dt} = -kXY$.

"The first is the reciprocal of the volume taken up by the impurities" => translates to: X, say, is $\frac{1}{V(t)}$

"The second is the amount of mass present in the filter" => translates to: Y, say, is $I(t)$.

Hence the ODE is

$$\frac{dR_{out}}{dt} = -k\,\frac{I(t)}{V(t)} = -k\,\frac{10t}{10^{-4}V_{filter}e^{kt}} = -K\,10^{5}V_{filter}^{-1}te^{-0.05t}.$$

- The initial value explicitly given $R_{out}(t_0) = 10$.
- The IVP is then

$$\begin{cases} \frac{dR_{out}}{dt} = -k\,10^{5}V_{filter}^{-1}te^{-0.05t} & t_0 < t < t_1 \\ R_{out}(t_0) = 10 \end{cases}$$

Extra credit:

Term	Units
$I(t)$	mg/s
$V(t)$	cm^3/s
$I(t)/V(t)$	mg/cm^3
dR_{out}/dt	cm^3/s^2
k	cm^6/mg s^2

$$\Rightarrow \frac{\cancel{cm^6}}{\cancel{mg}\,s^2}\cdot\frac{\cancel{mg}}{\cancel{cm^3}} = \frac{cm^3}{s^2}.$$

Assignment 2

Due: 2/1/18

All work must be shown in your solutions, even if it is not explicitly asked for you to explain. Use your own paper to turn in solutions. Only neatly presented solutions will be graded.

Note that the following model is likely entirely fictional. The point of the problem is for practice in ODE modeling something that seems reasonable, and gaining insight into modeling. Also, to be fair, all models are inaccurate to some extent.

Problem 1

(3pts per part)

Consider the cylindrical filtered pipe drawn on the next page. ① Air is pumped through it at a rate of $R_{in} = R_{out} = 10\text{cm}^3/\text{s}$. The purpose of the filter is to filter out impurities in the air, and it is known that the impurities have a concentration of 1mg/cm^3 in the air. At $t = 0$ seconds, the filter is assumed to ③ be clean; i.e., contains no impurities initially. Moreover, the impurities have a volume that is negligible with respect to the volume of air (i.e., we can assume that the impurities do not make up a considerable volume of the air-impurity mixture). As the impure air passes through the filter, all of the impurities are captured in the filter, and thus clean air exits the filter. ②

(a) Construct an IVP which models the mass of impurities $I(t)$ captured in the filter after t seconds. The units of the ODE should be mg/s. *Hint: by assumption, the filter is initially clean and therefore initially has no mass of impurities captured.* Now solve the IVP using separation of variables (answer: $I(t) = 10t$).

The filter is a cylinder of length 0.1cm and radius 0.3cm. Let V_{filter} be the volume of the filter. As the impurities are captured in the filter, after t seconds they take up a volume $V(t)$ of the filter, and the entire volume of the filter may be replaced by the impurities; i.e., $V(t) = V_{\text{filter}}$ is allowed for some time t. ① Assume that the rate at which the impurities take up volume in the filter is proportional to the current volume of impurities present in the filter. (The constant of proportionality can be understood as a constant reflecting the physical nature of the impurities or filter.) We also make the assumption that at $t = 0$, the impurities take up 0.01% of the filter's volume[1]. ②

(b) Construct an IVP which models the volume $V(t)$ buildup of the impurities. The units of the ODE should be cm^3/s. Let κ be the constant of proportionality. Now solve the IVP using separation of variables (answer: $V(t) = 10^{-4}V_{\text{filter}}e^{\kappa t}$). (Note that this solution only makes sense until $V(t) = V_{\text{filter}}$.) You don't have to compute V_{filter}.

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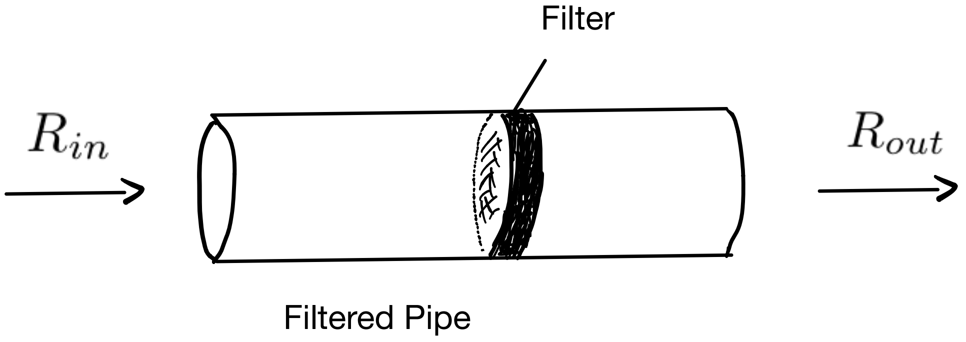
[1] While this assumption may seem contradictory to the filter being initially clean, it is an assumption we make to make the model work. Indeed, as more volume is captured, we might expect impurities are captured faster as a result of the filter "clogging" up. Hence, an exponential type growth is a reasonable model. However, an exponential model cannot have an initial condition of $V(0) = 0$ since the exponential model can never be zero, unless it is identically zero.

Only after the impurities take up 50% of the volume of the filter does the rate R_{out} at which air is pumped out slow down considerably. Assume that $\kappa = 0.05$, and let t_0, t_1 be the times for which $V(t_0) = 0.5V_{\text{filter}}$, and $V(t_1) = V_{\text{filter}}$. Since the purpose of the filter is so that clean air flows out of the pipe, we want the pipe to also function properly, and so predicting R_{out} as a function of time is desirable[2]. Suppose we know that the rate at which R_{out} changes is jointly and negatively proportional to two things[3]. The first is the reciprocal of the volume taken up by the impurities. The second is the amount of mass present in the filter after t seconds. Let $K > 0$ be the constant of proportionality.

(c) Construct an IVP which models the rate R_{out} at which air is pumped out for $t_0 \leq t \leq t_1$ (any other time is irrelevant based on the assumptions made). The units should be cm^3/s^2. *Hint: the initial value is given by $R_{out}(t_0) = 10cm^3/s$.* You don't have to solve for t_0, t_1.

Extra Credit: 2pt: (*To get credit: an honest attempt should be made apparent.*)
What units should K be? Explain why the assumptions made in part (b) and (c) might be reasonable. *Hint: What does the ratio mass/volume mean and how might this affect the rate at which air exits the pipe? Why should we assume negative proportionality?*



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[2] E.g., we might want to know how long the filter will last until the pipe-filter system loses its functionality.

[3] Recall that X is negatively jointly proportional to Y and Z if $X = -KYZ$ for some $K > 0$.