Section 2.2
Separable Equations

Concepts From Homework:

- Solving First Order ODEs/IVPs by separation of variables
  - when ODE is in differential form $P(x,y)\,dx + Q(x,y)\,dy = 0$
  - When ODE is in usual form
- Produce solutions explicitly, implicitly, and by an integral.
- Constant solutions and losing solutions.

TODO: Make sure you review your integration techniques.

Lecture Notes

Δ Some Preliminaries

Definition (ODE in differential Form)
If we may write the ODE
$$F(x,y,y')=0$$
in the form
$$P(x,y)\,dx + Q(x,y)\,dy,$$
we say we have placed the ODE in differential form.

- Recall that if x and y are related by $y=f(x)$, the dy is given by
$$dy = f'(x)\,dx$$
- Though imprecise, we think of obtaining dy from $\frac{dy}{dx}=f'(x)$ by "multiplying" through by dx:
$$dy = \frac{dy}{\cancel{dx}}\cdot \cancel{dx} = f'(x)\,dx.$$

- Recall that sometimes the best we can do in representing a function is by an implicit equation:

  In $\arctan(e^{y+x}) = \sqrt{y} + x^y$, we cannot solve for x in terms of y.
  But this equation implicitly defines a function $y=y(x)$.

Definition (Implicit/Explicit Solutions)
If a solution to an ODE is obtained by an implicit equation
$$G(x,y)=0,$$
we call this equation an implicit solution. If the solution may be written in the form $y=f(x)$ for some explicit function f, we call $y=f(x)$ an explicit solution.

- Let $\frac{dy}{dx}=f(x,y)$ be an ODE. Recall that if $\frac{dy}{dx}=0$ for x in an interval, then y is constant on that interval.

Definition (Constant Solution)
Suppose a solution $y=y(x)$ to $y'=f(x,y)$ is obtained by setting $y'=f(x,y)=0$. Then y is called a constant solution.

Ex If $y'=\cos(y)$, then the constant solutions are given by $y'=\cos(y)=0$; i.e., the constant functions $y=k+\pi/2$, k an integer.
~ Used in last WebAssign 2.2 problem.

Δ Introduction

- Goal: Learn our first method to solving certain types of ODEs.
- (*) Common theme in DEs: How we solve an ODE depends on the "type" of DE.
- The type of DE we solve here are "separable ODEs".
- Our main tool will be integration. So Review it!

(*) DE type $\longleftrightarrow$ Method of Analysis

Ex

Consider the ODE
$$\frac{dy}{dx} = xy.$$
Let's pretend we can multiply through by dx:
$$dy = \cancel{dx}\frac{dy}{\cancel{dx}} = xy\,dx.$$
Then
$$\frac{1}{y}\,dy = x\,dx.$$
Since we have divided by y, we must here on assume $y \neq 0$.

Now hit both sides with an integral:
$$\ln|y| + c_1 = \int \frac{1}{y}\,dy = \int x\,dx = \frac{1}{2}x^2 + c_2$$
Combining the constants of integration:
$$\ln|y| = \frac{1}{2}x^2 + c_2 - c_1 = \frac{1}{2}x^2 + C$$
Solving for $|y|$:
$$|y| = e^{\frac{1}{2}x^2 + C} = Ae^{\frac{1}{2}x^2},\ A > 0$$
Hence
$$y = \pm Ae^{\frac{1}{2}x^2},\ A > 0$$
are solutions.

Absorbing the sign into A:
$$y = Ae^{\frac{1}{2}x^2},\quad A \neq 0.$$
But clearly $y = 0$ solves $y' = xy$, so we may take A to be any real number.
Recall above we divided by zero, which is why we had the restriction $A \neq 0$ at first.
Let's check $y = Ae^{\frac{1}{2}x^2}$ is a solution:
$$y' = \frac{1}{2}A\,2x\,e^{\frac{1}{2}x^2} = xAe^{\frac{1}{2}x^2} = xy \checkmark$$
So that multiply by dx trick worked!

- Observe that the method above worked so well because we were able to separate the variables: all things with an x in it to one side, and things with y's to the other side.

Definition (Separable ODE)

Suppose we have the ODE
$$(*)\quad \frac{dy}{dx} = f(x,y).$$
If we may separate the variables of f, i.e., we may find functions $g(x)$ and $h(y)$ so that
$$\frac{dy}{dx} = f(x,y) = g(x)\,h(y)$$
then we call (*) a separable ODE.

- In the field or other course you may find the need to learn new mathematical tools and techniques. To really understand a mathematical tool, it is sometimes best to see how the tool is derived. Seeing the derivation gives you a better understandin of the tool "under the hood".
  - In practice it may not be clear which method to use. Seeing the solution method derived may help
  - Let us see how separating variables may solve separable ODEs.
- The following method is called Separation of variables

Separation of variables:

- Suppose the ODE $\frac{dy}{dx} = f(x,y)$ is separable.
- So there are $g(x), h(y)$ such that
$$\frac{dy}{dx} = f(x,y) = g(x)\,h(y).$$
- Assume $h(y) \neq 0$ and divide:
$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$$
- Suppose $y = F(x)$ is a solution. Then
$$\frac{1}{h(F(x))}F'(x) = g(x)$$
- Integrating gives
$$\int \frac{1}{h(F(x))}F'(x)\,dx = \int g(x)\,dx$$
- Doing the change of variable $y = F(x)$, $dy = F'(x)dx$
$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$
- Then if $H(y)$, $G(x)$ antiderivatives of $\frac{1}{h(y)}$, $g(x)$:
$$H(y) + c_1 = G(x) + c_2$$
- Or, writing $c = c_2 - c_1$:
$$H(y) = G(x) + c.$$

The boxed in definition is to help understand separation of variables is and how it works. You need not know how to do this proof.

- If all went well, this defines $y = y(x)$ implicitly. Hence we obtain a 1-parameter family of implicit solutions.

- Note that we assumed $h(y) \neq 0 \Rightarrow$ we restrict to intervals where $h(y) \neq 0$.
- Note that separation of variables produces at best a 1-parameter family of solutions. So this method does not see singular solutions. So more analysis is always needed to find all solutions, if more solutions exist.

Ex Solve the following ODEs:

(a) $(y+1)dx - (x+1)^2 dy = 0$

(b) $(x^2+1)\frac{dy}{dx} = e^y$

(c) $y' = 2\sqrt{y}$

(d) $\begin{cases} y' = \frac{xy+x}{xy-y} \\ y(0) = 0 \end{cases}$

(e) $\begin{cases} y' = e^{x^3} \\ y(0) = 1 \end{cases}$

(a) $(y+1)dx - (x+1)^2 dy = 0$

$$(y+1)dx = (x+1)^2 dy$$

$$\frac{dx}{(x+1)^2} = \frac{dy}{y+1}$$

$$\int \frac{dx}{(x+1)^2} = \int \frac{dy}{y+1}$$

$$-(x+1)^{-1} + c = \ln|y+1|$$

$$|y+1| = e^{-(x+1)^{-1}+c}$$
$$= e^c e^{-(x+1)^{-1}}$$
$$= A e^{-(x+1)^{-1}}, \quad A > 0$$

$\therefore$

$$y+1 = Ae^{-(x+1)^{-1}}, \quad A \neq 0$$

But

$$(y+1)dx - (x+1)^2 dy = 0 \iff (y+1) - (x+1)^2 \frac{dy}{dx} = 0$$

and so $y = -1$ is a solution. So $A = 0$ is okay.

Hence

$$y = Ae^{-(x+1)^{-1}} - 1, \quad A \in \mathbb{R}$$

is a one parameter family of solutions.

(b)

$$(x^2+1)\frac{dy}{dx} = e^y$$

$$e^{-y} dy = \frac{dx}{x^2+1}$$

$$\int e^{-y} dy = \int \frac{dx}{x^2+1}$$

$$-e^{-y} = \arctan(x) + c, \quad c \in \mathbb{R}$$
$$e^{-y} = -\arctan(x) + c, \quad c \in \mathbb{R}$$

This only holds where $-\arctan(x) + c > 0$ (since $e^{-y} > 0$ is always true)

Supposing this holds, then

$$y = -\ln(-\arctan(x) + c)$$

(c) $y' = 2\sqrt{y}$ Observe $y = 0$ is a solution.

$$\frac{dy}{dx} = 2\sqrt{y}$$

$$y^{-\frac{1}{2}} dy = 2dx$$

$$\int y^{-\frac{1}{2}} dy = \int 2dx$$

$$2y^{\frac{1}{2}} = 2x + c$$

$2y^{\frac{1}{2}} = 2x + C$

$4y = (2x+c)^2 \Rightarrow y = \frac{1}{4}(2x+c)^2.$

So $y=0$ is singular: it cannot be obtained from $\frac{1}{4}(2x+c)^2$.

(d) $y' = \dfrac{xy+x}{xy-y}$

$$\frac{dy}{dx} = \frac{xy+x}{xy-y}$$

$(xy-y)\,dy = (xy+x)\,dx$

$(1-x^{-1})\,dy = (1+y^{-1})\,dx$ (by dividing by xy)

$$\frac{dy}{1+y^{-1}} = \frac{dx}{(1-x^{-1})}$$

$$\frac{y}{y}\,\frac{dy}{1+y^{-1}} = \frac{x}{x}\,\frac{dx}{1-x^{-1}}$$

$$\frac{y\,dy}{y+1} = \frac{x\,dx}{x-1}$$

$$\therefore \int \frac{y\,dy}{y+1} = \int \frac{x\,dx}{x-1}$$

But

$$\int \frac{y}{y+1}\,dy = \int \frac{y+1-1}{y+1}\,dy = \int \frac{y+1}{y+1} - \frac{1}{y+1}\,dy = \int 1 - \frac{1}{y+1}\,dy = y - \ln|y+1| + C$$

$$\int \frac{x\,dx}{x-1} = \int \frac{x-1+1}{x-1}\,dx = \int \frac{x-1}{x-1} + \frac{1}{x-1}\,dx = \int 1 + \frac{1}{x-1}\,dx = x + \ln|x-1| + C$$

$$\therefore \quad y - \ln|y+1| = x + \ln|x-1| + C.$$

By $y(2)=1$, we have

$y(2) - \ln|y(2)+1| = 2 + \ln|2-1| + C$

$1 - \ln|1+1| = 2 + C$

$C = -1 - \ln 2$

So $y - \ln|y+1| = x + \ln|x-1| - 1 - \ln 2$ is the implicit solution.

This is an example of an implicitly defined sol.
By implicit differentiation you can show $y=y(x)$ defined
by this eq. satisfies the ODE.

(e) $y' = e^{x^3}$

Initial condition: $y(0)=1$. FTOC gives desired particular solution:

$$\int_0^x y'\,dx = \int_0^x e^{t^3}\,dt \quad \text{(use dummy variable } t\text{)}$$
$$y\Big|_0^x = \int_0^x e^{t^3}\,dt \quad \text{(use FTOC)}$$

$$y(x) - y(0) = y(x) - 1 = \int_0^x e^{t^3}\,dt$$

So $y = 1 + \int_0^x e^{t^3}\,dt$ is the desired solution.

The point is that sometimes separation of variables ends in an integral we can at best numerically solve.

Hint for 2.2.7:
Rewrite the ODE by dividing through by dx!

$$2x\sin^2(y)\,dx - (x^2+17)\cos(y)\,dy = 0$$

$$\Rightarrow \quad 2x\sin^2(y) - (x^2+17)\cos(y)\frac{dy}{dx} = 0$$

To get constant sols, set $\frac{dy}{dx} = 0$:

$$2x\sin^2(y) - (x^2+17)\cos y \cdot 0 = 2x\sin^2(y) = 0.$$

So we need

$$\sin^2(y) = 0$$

When does this happen?