Section 1.2

Concepts from homework

Initial value problem – given family of solutions + initial condition, find corresponding particular solution
– find largest interval this particular solution satisfies the ODE

IVP for order 2 ODE – given 2-parameter family of solutions, two initial conditions $x(t_0) = x_0$, $x'(t_0) = x_1$, find appropriate c_1, c_2.

Existence and Uniqueness – Given ODE, find appropriate rectangles to apply existence and uniqueness theorem.

Notes: In problem 5b), note that y being a solution to $y' = 1 + y^2$
$\Rightarrow$ y is continuous on its interval of definition $\Rightarrow$ since y is discontinuous on $(-2, 2)$, $(-2, 2)$ cannot be an interval of definition.

Lecture Notes

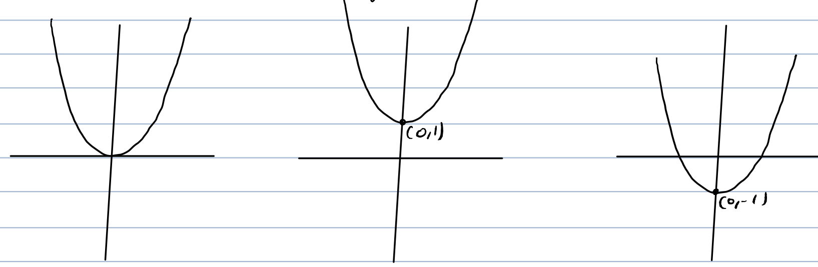
Δ Introduction

Consider the first order ODE $y' = 2x$. It has as a 1-parameter family of solutions $f_c(x) = x^2 + c$:

$$f_c'(x) = (x^2 + c)' = 2x$$

(The largest interval of definition is $(-\infty, +\infty)$.)

For different values of c, we get different "solution curves":



$c = 0$ $\qquad$ $c = 1$ $\qquad$ $c = -1$

It's not hard to see that only one of the curves $f_c(x) = x^2 + c$ passes through the point $(0, b)$ for each $b \in \mathbb{R}$:

$$f_c(0) = b \Rightarrow 0^2 + c = b \Rightarrow c = b.$$

I.e., only $f_b(x) = x^2 + b$ passes through $(0, b)$.

Similarly, only one of the curves $f_c(x) = x^2 + c$ passes through the point $(1, b)$ for each $b \in \mathbb{R}$:

$$f_c(1) = b \Rightarrow 1^2 + c = b \Rightarrow c = b - 1$$

I.e., only $f_{b-1}(x) = x^2 + b - 1$ passes through $(1, b)$.

It turns out that if we impose the conditions

$$(*) \quad \begin{cases} y' = 2x & x \in \mathbb{R} \\ y(x_0) = b \end{cases}$$

for some "initial" point x_0 and value b, there is exactly one solution to $(*)$, namely

$$f_c(x_0) = b \Rightarrow x_0^2 + c = b \Rightarrow c = b - x_0^2$$

$\Rightarrow f_{b-x_0^2}(x) = x^2 + b - x_0^2$ is the only solution to $y' = 2x$ satisfying $y(x_0) = b$.

Note that the uniqueness property observed above is not guaranteed.

Definition (Initial Value Problem)
The problem of finding an n-times differentiable function subject to the conditions

$$\begin{cases} \frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}), \quad x \in I \\ y(x_0) = y_0,\ y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}, \end{cases}$$

where $x_0 \in I$, I an interval, and f a function, is called an n-th order initial value problem (IVP).

- Note: An nth order problem will usually have an n-parameter family of solutions, and hence need n conditions to specify the corresponding constants.
- Note: We typically think of x_0 as the "starting point." This is especially true if the independent variable is time and $t_0 = 0$.

Δ So how to solve an nth order IVP? Typically we follow

1) Find n-parameter family of solutions.
2) Take appropriate derivatives and apply initial conditions to obtain n equations of n unknowns
(sometimes this step is simplified)
3) Solve system.

Ex The first order linear ODE $y'' - \sin(x) = 0$ has as a two-parameter family of solutions

$$f_{c_1, c_2}(x) = c_1 x + c_2 - \sin(x).$$

Find a solution to the second order IVP

$$\begin{cases} y'' - \sin(x) = 0 \\ y(0) = 2,\ y'(0) = 0 \end{cases}$$

What is the largest interval I a solution can be defined on?

Step 1): This is given.

Step 2): We find f'_{c_1,c_2}: $f'_{c_1,c_2}(x) = (c_1x + c_2 - \sin(x))' = c_1 - \cos(x)$

$$y(0)=2 \Rightarrow \quad f_{c_1,c_2}(0) = c_1 \cdot 0 + c_2 - \sin(0) = c_2 = 2$$
$$y'(0)=0 \Rightarrow \quad f'_{c_1,c_2}(0) = c_1 - \cos(0) = c_1 - 1 = 0 \Rightarrow c_1 = 1.$$

Step 3): The work above solved for c_1, c_2 already.

Therefore a solution to the IVP is

$$f_{1,2}(x) = x + 2 - \sin(x).$$

This function is defined everywhere and satisfies the ODE everywhere. Hence $I = \mathbb{R}$.

Ex

Consider the ODE $yy' + nx = 0$, where n is any integer.
An one-parameter family of solutions is given by

$$f_{c_1}(x) = \sqrt{c_1 - nx^2}.$$

Find a solution to the IVP

$$\begin{cases} yy' + nx = 0 \\ y(-1) = 0 \end{cases}.$$

Give the largest interval I such a solution may be defined.

Let's check f_{c_1} is indeed a solution:

$$f'_{c_1}(x) = (\sqrt{c_1 - nx^2})' = \frac{1}{2} \frac{-2nx}{\sqrt{c_1 - nx^2}} = \frac{-2nx}{\sqrt{c_1 - nx^2}}$$

So

$$f_{c_1}(x) f'_{c_1}(x) + nx = \frac{-nx}{\sqrt{c_1 - nx^2}} \sqrt{c_1 - nx^2} + nx$$
$$= -nx + nx$$
$$= 0.$$

Thus f_{c_1} is indeed a solution.
We now find c_1 by using $y(-1) = 0$:

$$f_{c_1}(-1) = 0 \Rightarrow \quad 0 = \sqrt{c_1 - n(-1)^2} = \sqrt{c_1 - n} \Rightarrow c_1 - n = 0 \Rightarrow c_1 = n.$$

Therefore a solution to the IVP is
$$f_n(x) = \sqrt{n - nx^2}.$$

We now find I. First note that if $n=0$, then $f_0(x) = 0$. Thus the solution f_0 is defined everywhere. Now suppose $n \neq 0$. In order for f_n to be defined, we must have

$$n - nx^2 \geq 0 \iff n \geq nx^2 \iff 1 \geq x^2 \iff 1 \geq x \geq -1 \iff x \in [-1, 1]$$

Since we only work in open intervals, we have $I = (-1, 1)$.
Therefore
$$I = \begin{cases} (-1,1) & \text{if } n \neq 0 \\ \mathbb{R} & \text{if } n = 0 \end{cases}.$$

Ex (IVP with Context)

Suppose we know that the acceleration of a particle is given by the IVP

$$\begin{cases} \dfrac{d^2x}{dt^2} = \sin t \quad \text{on } (-\pi/2, \pi/2) \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

A two-parameter family of solutions to this ODE is given by

$$f_{c_1,c_2}(t) = -\sin(t) + c_1 t + c_2.$$

Find the displacement of the particle at time $t = \pi/4$.

First we use the initial conditions to find the appropriate solution:

$$x(0) = 0 \implies f_{c_1,c_2}(0) = -\sin(0) + c_1 \cdot 0 + c_2 = c_2 = 0$$
$$x'(0) = 0 \implies f'_{c_1,c_2}(0) = -\cos(0) + c_1 = -1 + c_1 = 0 \implies c_1 = 1$$

Therefore $x(t) = f_{1,0}(t) = -\sin(t) + t$ is the desired solution.

We may now predict where the particle will be at time $t = \pi/4$:

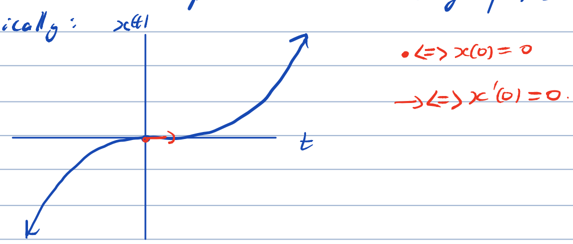
$$x(\pi/4) = -\sin(\pi/4) + \pi/4 = -\sqrt{2}/2 + \pi/4.$$

We note that the initial conditions

$$\begin{cases} x(0) = 0 \\ x'(0) = 0 \end{cases}$$

specify the initial displacement and velocity of the particle.

Graphically:



Δ Does an IVP always have a solution? When is it unique?

Quick answer: No and not usually.

However, there are conditions we may impose on the ODE to guarantee existence and uniqueness.

Theorem (Existence and Uniqueness)

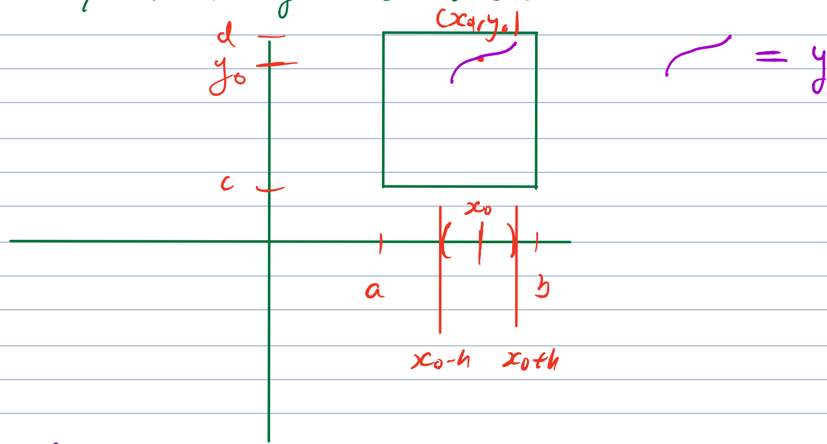
Consider the first order IVP

$$\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases} \qquad (\star)$$

Let R = rectangular region in xy-plane given by $a \le x \le b$, $c \le y \le d$

Suppose
- R contains the point (x_0, y_0) in its interior
- $f(x,y)$ continuous on R
- $\frac{\partial f}{\partial y}$ continuous on R

Then there is an interval $I_0 = (x_0 - h, x_0 + h)$, $h > 0$ contained in $[a,b]$, and a unique function y that solves $(\star)$ on I_0.



- $\frac{\partial f}{\partial y}$ = derivative of f with respect to y. Eg., if $f(x,y) = x + xy^2$, then
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x + xy^2) = 0 + 2xy = 2xy.$$

- There is a more general version of this theorem. We will come back to it later.

Ex Determine the region in the xy-plane so that

$$\begin{cases} \sqrt{y}\, y' + x = 0 \\ y(x_0) = y_0 \end{cases},$$

where (x_0, y_0) is any point in the region, has a unique solution.

This question translates to:

> Find the largest region so that for any point in this region, we may apply the Existence and uniqueness theorem on any rectangle in the region.

So we need to determine the appropriate f by putting $\sqrt{y}\, y' + x = 0$ into the form
$$\frac{dy}{dx} = f(x,y).$$

So we solve for $\frac{dy}{dx}$:

$$\sqrt{y}\, y' + x = 0 \implies y' = -\frac{x}{\sqrt{y}}.$$

Hence $f(x,y) = -\frac{x}{\sqrt{y}}$, and so $\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}\left(\frac{-x}{\sqrt{y}}\right) = \frac{1}{2} x y^{-3/2}$.
But f and $\frac{\partial}{\partial y} f$ are continuous anywhere $y > 0$. Hence, if $y_0 > 0$, then there is a unique solution.

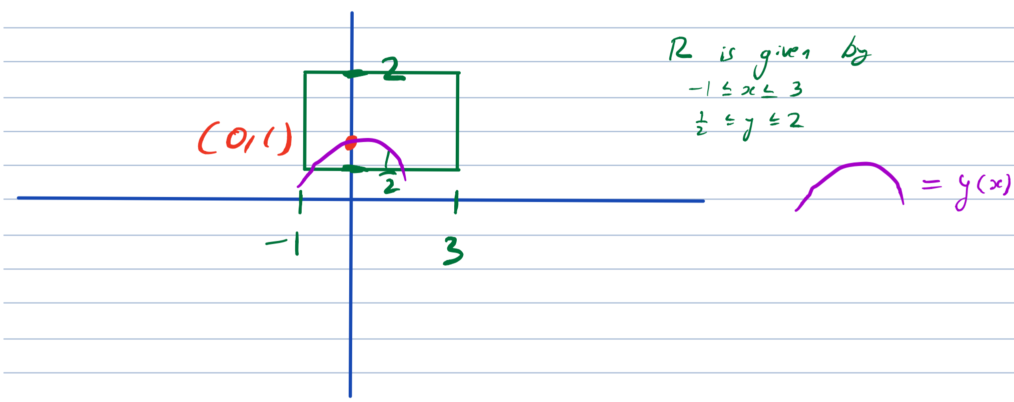
For example, let $x_0 = 0$, $y_0 = 1$. Then the IVP is

$$\begin{cases} \sqrt{y}\, y' + x = 0 \\ y(0) = 1 \end{cases}$$

Here we still have

$$f(x,y) = \frac{-x}{\sqrt{y}}, \quad \frac{\partial}{\partial y} f(x,y) = \frac{1}{2} x y^{-3/2}$$

which are continuous where $y \neq 0$.
An example rectangle R to apply the theorem to is given below:



In conclusion, the desired region is given by $y > 0$, any x.

Ex (Of non-uniqueness)
The IVP

$$\begin{cases} y' = x y^{1/2} \\ y(0) = 0 \end{cases}$$

has two solutions.
Indeed, both

$$y_1 = 0$$

$$y_2 = \frac{1}{16} x^4$$

solve this IVP:

$$y_1' = 0 \qquad y_2' = \frac{1}{4} x^3 = x \cdot \left(\frac{x^4}{16}\right)^{1/2} = x y^{1/2}$$

$$y_1(0) = 0 \qquad y_2(0) = 0$$

So what's the deal? Observe that for this IVP, $f(x,y) = x y^{1/2}$, and that

$$\frac{\partial}{\partial y} f(x,y) = \frac{1}{2} \frac{x}{\sqrt{y}},$$

which is not continuous at $y = 0$. Therefore, we may not apply the existence and uniqueness theorem to the point $(0,0)$.