Section 1.1

Concepts from homework

- ✓ Order of ODE — given an ODE, find its order, differentiate between $\frac{d^2y}{dx^2}$, $\left(\frac{dy}{dx}\right)^2$
- ✓ Linearity — Compare given ODE with $\sum a_n(x) f^{(n)}(x)$ to determine linearity
- ✓ Verifying a solution — given a solution, check via computation it satisfies ODE
- ✓ Interval of definition
- ~ Explicit solution
- ✓ Domain of a function versus domain of solution — by direct comparison
- ✓ Family of solutions — check a family satisfies an ODE
- ✓ Integral defined function — check an integral defined function is a solution

Just use FTOC: $F(x) = \int_a^x g(t)\,dt \Rightarrow F'(x) = g(x)$

Notation: $x \in I \Rightarrow x$ is in the interval I. $\mathbb{R}$ = all real numbers.

Lecture Notes

△ Introduction

Before we can talk about why one should take this course, we need to learn some basic terminology and ideas regarding ordinary differential equations (ODE).

Hence we begin by learning the language of ODEs. Motivation, while important, will have to wait.

△ What is a differential equation?

Informally: Without context: an equation with derivatives and whose solutions are functions.

E.g., $y' = y$ with solution $y = ce^x$.

With context: an equation that models the evolution of some system.

E.g., let t = time, $V(t)$ = voltage across some capacitor.
Suppose we know that the voltage is described by the model:

$$\frac{d^2V}{dt^2} = -V$$

Then the voltage may take the form $V(t) = \sin(t)$. (other functions work too)

The Idea: We construct an ODE that models how a system evolves. The solution will be a property of the system we wish to compute.

E.g., $V(t) = \sin(t)$ allows us to compute voltage at any time.

Formally:

Definition (Ordinary Differential Equation)

Any equation of the form

$$F(x, y, y', y'', \dots, y^{(n)}) = 0,$$

where $y^{(n)} := \frac{d^n y}{dx^n}$,

is called an ordinary differential equation (ODE).

In English: an ODE is an equation involving an independent variable x, a dependent variable $y = y(x)$, and derivatives $y, y', y'', \dots, y^{(n)}$.

Note: When we write $F(x, y, y', \dots, y^{(n)}) = 0$, we assume this ODE depends on $y^{(n)}$ in a non-trivial way, but F may depend on the other derivatives in a trivial way.

Ex The following differential equations are ODEs:

Note that we call equations like $y'' = e^x$ an ODE, even though it is not in the form $y'' - e^x = 0$.

(a) $\frac{dy}{dx} = e^x - y$

$F(x, y, y') = y' - e^x + y$

(b) $\frac{d^2y}{dx^2} = e^x$

$F(x, y, y', y'') = y'' - e^x$

(c) $\frac{d^2y}{dx^2} = \cos(x)$

$F(x, y, y', y'') = y'' - \cos(x)$

(d) $\arctan(y''') = xy$

$F(x, y, y', y'', y''') = \arctan(y''') - xy$

Ex What is the apparent difference between the ODEs

$$y' = \frac{dy}{dx} = f(x) \quad \text{and} \quad y'' = \frac{d^2y}{dx^2} = f(x)?$$

For $y' = f(x)$, the highest appearing derivative is y'. For $y'' = f(x)$, it is y''. Equivalently, if $F_1(x, y, y') = y' - f = 0$, $F_2(x, y, y', y'') = y'' - f = 0$ we see that F_1 depends on at most y', and F_2 on at most y''.

Ex (with context)

Let $x = x(t)$ give the displacement of a particle at time t.

Recall

$$\frac{dx}{dt} = \text{velocity}, \quad \frac{d^2x}{dt^2} = \text{acceleration}.$$

Then the two ODEs

$$\frac{dx}{dt} = f(t) \quad \text{and} \quad \frac{d^2x}{dt^2} = f(t)$$

describe two different things. $\frac{dx}{dt} = f(t) \Rightarrow f(t)$ describes the velocity

$\frac{d^2x}{dt^2} = f(t) \Rightarrow f(t)$ describes the acceleration.

So we see that the derivatives present in an ODE may represent different things. Thus we introduce the following definition.

Definition (Order)
In an ODE, the highest order derivative present is called the order of the ODE; i.e., if

$$F(x, y, y', y'', \dots, y^{(n)}) = 0$$

is an ODE, then the order of this ODE is n (recall the convention in the definition above).

Ex Find the order of the following ODEs.

(a) $y' = e^{x}$

(b) $y''' - y\,y^{(4)} = x\cos(y'')$

(c) $\frac{d^3x}{dt^3} - \frac{d^7x}{dt^7} = \cos t \cdot \frac{d^{10}x}{dt^{10}}$

(d) $\left(\frac{dy}{dx}\right)^{10} = \frac{d^5y}{dx^5}$

(a) $y^{\boxed{1}} = e^{x} \Rightarrow$ highest order is $y \Rightarrow$ order $=1$

Equivalently: $F(x, y, y') = y' - e^{x} \Rightarrow n = 1$

(b) $y''' - y\,y^{\boxed{(4)}} = x\cos(y'') \Rightarrow$ highest order is $y^{(4)} \Rightarrow$ order $= 4$

Equivalently: $F(x, y, y', y'', y''', y^{(4)}) = y''' - y\,y^{(4)} - x\cos(y'') = 0 \Rightarrow n = 4$

(c) $\frac{d^3x}{dt^3} - \frac{d^7x}{dt^7} = \cos t \cdot \frac{d^{\boxed{10}}x}{dt^{10}} \Rightarrow$ highest order is $\frac{d^{10}x}{dt^{10}} \Rightarrow$ order $= 10$

Equivalently: $F(t, x, x', \dots, x^{(10)}) = x''' - x^{(7)} - \cos t\, x^{(10)} = 0 \Rightarrow n = 10$

(d) $\left(\frac{dy}{dx}\right)^{10} = \frac{d^{\boxed{5}}y}{dx^5} \Rightarrow$ highest order is $\frac{d^5y}{dx^5} \Rightarrow$ order $= 5$

In (d), note that $\left(\frac{dy}{dx}\right)^{10}$ only has an order one derivative: it is the first order derivative $\frac{dy}{dx}$ raised to the tenth power.

Δ So now that we know what an ODE is, we need to know what it means to solve an ODE.

Ex
(a) Question: What does it mean to solve

$x^2 - 1 = 0$?

Answer: Find all x so to $x^2 - 1 = 0$ is numerically true:

So $x = \pm 1$

(b) Question: What does it mean to solve $x^2 + y^2 = 1$ for a continuous function $y = f(x)$?

Answer: Find a continuous function $y = f(x)$ so that

$$x^2 + f(x)^2 = 1$$

is numerically true for all x in the domain of f.

We know $f(x) = \sqrt{1-x^2}$ or $f(x) = -\sqrt{1-x^2}$ on $[-1,1]$ both are solutions.

Thus solutions to (a) or (b) are things which satisfy a condition specified by an equation.

(c) Question: What might it mean to solve $y' = y$ for a differentiable function $y = f(x)$?

Answer: Motivated by (a) and (b), we want to find a differentiable function f which satisfies

$$f'(x) = f(x)$$

for suitable x.

We give two answers: write

$$f(x) = e^x, \quad f_c(x) = ce^x, \; c \in \mathbb{R}.$$

Observe
$$f'(x) = (e^x)' = e^x = f(x) \qquad f_c'(x) = (ce^x)' = ce^x = f_c(x)$$

and these hold true for all $x \in \mathbb{R}$. We will call f a particular solution, and f_c a one-parameter family of solutions.

Note that f_c is somehow a more general solution than f since $f_1(x) = 1 \cdot e^x = e^x = f(x)$.

Definition (Solution to an ODE / trivial Solution)

Given the nth order ODE

$$F(x, y, y', \ldots, y^{(n)}) = 0 \qquad (*)$$

Note that we always talk about a solution on an interval

a solution to $(*)$ on an interval $I = (a,b)$ is a differentiable function f so that

$$F(x, f(x), f'(x), \ldots, f^{(n)}(x)) = 0$$

is numerically true for all x in (a,b). I is called the interval of definition. If $y = 0$ is a solution, it is called the trivial solution.

In English: a solution to an ODE is a function so that if you plug it into the equation, the equation holds true on the specified interval.

Note: we will get into this more later, but solutions are not always obtainable in a explicit form. If a solution is given by

$$y = f(x), \quad f \text{ an elementary function.}$$

for some function f, then this solution is called an explicit solution. However, such an expression is sometimes impossible. You may face two other cases:

- a solution may be given via an integral
- a solution may be given implicitly via an implicit equation

Ex Verify whether or not the given function satisfies the specified ODE on the specified interval:

(a) $f(x) = \sin(x)$, $\frac{d^2y}{dx^2} = -y$ on $I = \mathbb{R}$

(b) $f(x) = c + \frac{1}{x}$, $\frac{dy}{dx} = -\frac{1}{x^2}$ on $I = (0, +\infty)$

(c) $f(x) = \frac{1}{c-x}$, $y' - y^2 = 0$ on $I = (0, 2)$ — what values of c are allowed?

Idea: just substitute $y = f(x)$ into the given ODE and inspect whether or not it holds true on the given interval.

(a) We find

$$f''(x) = (\sin(x))'' = (\cos(x))' = -\sin(x) = -f(x)$$

Hence, $f''(x) = -f(x)$, clearly holds for all $x \in \mathbb{R}$.

Equivalently, here we may take $F(x, y, y', y'') = y'' + y = 0$. Then

$$F(x, f(x), f'(x), f''(x)) = f''(x) - f(x) = -\sin(x) + \sin(x) = 0$$

holds true for all $x \in \mathbb{R}$.

Thus $f(x) = \sin(x)$ is a solution. $I = \mathbb{R}$ is the interval of definition.

(b) $f(x) = c + \frac{1}{x} \Rightarrow f'(x) = \frac{d}{dx}\left(c + \frac{1}{x}\right) = \frac{d}{dx} c + \frac{d}{dx}\frac{1}{x} = 0 - \frac{1}{x^2} = -\frac{1}{x^2}$.

Hence

$$f'(x) = -\frac{1}{x^2}$$

and this is true for all $x \in (0, +\infty)$ (and $x \in (-\infty, 0)$, but not when $x = 0$.)

Equivalently, here $F(x, y, y') = y' + \frac{1}{x^2} = 0$, and

$$F(x, f(x), f'(x)) = (f(x))' + \frac{1}{x^2} = -\frac{1}{x^2} + \frac{1}{x^2} = 0.$$

So f is a solution, and $I = (0, +\infty)$ is the interval of definition.

(c) Here $f(x) = \frac{1}{c-x}$, which is undefined at $x = c$. Hence, if $c \in (0, 2)$, then $f'(x) - (f(x))^2 = 0$ cannot possibly hold.

So we must have $c \notin (0, 2)$ (i.e., $c \leq 0$ or $c \geq 2$.)

We check f is indeed a solution:

$$f'(x) = \left(\frac{1}{c-x}\right)' = -\frac{1}{(c-x)^2} = -\left(\frac{1}{c-x}\right)^2 = -(f(x))^2,$$

hence

$$f'(x) + (f(x))^2 = 0$$

is true on $(0,2)$ whenever $c \notin (0,2)$.

Equivalently, here $F(x,y,y') = y' + y^2 = 0$, and

$$F(x, f(x), f'(x)) = f'(x) + (f(x))^2 = -\left(\frac{1}{c-x}\right)^2 + \left(\frac{1}{c-x}\right)^2 = 0$$

whenever $c \notin (0,2)$.

Here f is a solution on $(0,2)$ whenever $c \notin (0,2)$.
Observe the the possible interval of definition depends on the definition.

Ex A solution to an ODE is on an interval, nothing more. So the domain of a function and its interval of definition to an ODE may differ.
E.g., $f(x) = 1/x$ solves $xy' + y = 0$ on $(-\infty, 0)$, $(0, +\infty)$, and these are the largest possible domains of definition. Yet, f may be defined on $(-\infty,0)\cup(0,+\infty)$, which is not an interval.

Δ How general can a solution be?

Ex The two expressions

$$f(x) = \sin(x) \quad \text{and} \quad f_{c_1,c_2}(x) = c_1\cos(x) + c_2\sin(x)$$

are both solutions to the second order ODE

$$y'' = -y \quad \text{on } I = \mathbb{R}$$ (Here we may take $F(x,y,y',y'') = y'' + y$)

$$f''(x) = -\sin(x) = f(x), \qquad f''_{c_1,c_2}(x) = -c_1\cos(x) - c_2\sin(x) = -f_{c_1,c_2}(x)$$

Yet

$$f_{0,1}(x) = f(x).$$

So the expression f_{c_1,c_2} contains the information that f is a solution.
Hence the family $\{f_{c_1,c_2}\}$ defines a 2-parameter family of solution.

Logically,

$$f_{c_1,c_2} \text{ a solution to } y'' = -y \Rightarrow f \text{ a solution to } y'' = -y,$$

But the converse is not obvously true. In terminology below, a particular solution does not hint at what a parameterized solution should be in general, if one were to exist.

Ex The two expressions

$$f(t) = e^{At}, \quad f_c(t) = ce^{At}$$

satisfy the first order ODE

$$\frac{dy}{dt} = Ay \quad \text{on } \mathbb{R}. \quad (\text{Here we may take } F(t, y, y') = y' - Ay)$$

Indeed,

$$f'(t) = (e^{At})' = Ae^{At} = Af(t)$$

$$f_c'(t) = (ce^{At})' = Ace^{At} = Af_c(t)$$

We see that the family $\{f_c\}$ defines a 1-parameter family of solutions.

Definition (n-parameter family of solutions)
Given an nth order ODE

$$F(x, y, y', \ldots, y^{(n)}) = 0, \quad (\star)$$

n continuous parameters $c_1, \ldots, c_n$, and an n-parameter family of functions $\{f_{c_1, \ldots, c_n}\}$, if each $f_{c_1, \ldots, c_n}$ is a solution to $(\star)$, then the family is called an n-parameter family of solutions.

- We will almost always see that a parameterized family of solutions to an nth order ODE has n parameters.
- More generally, an n-parameter family of solutions to $(\star)$ is an equation

$$G(x, y, c_1, \ldots, c_n) = 0 \quad (\star\star)$$

such that if $f(x) = y$ is implicitly defined by $(\star\star)$, then f solves $(\star)$

- As we have seen in a previous example, note that the interval of definition for each $f_{c_1, \ldots, c_n}$ may depend on the parameters $c_1, \ldots, c_n$.

Ex Verify that $f_c(x) = \cot(x+c)$ defines a one-parameter family of solutions to the first order ODE

$$y' = -1 - y^2.$$

We find $\quad (\cot(x+c))' = \left(\frac{\cos(x+c)}{\sin(x+c)}\right)' = \frac{-\sin^2(x+c) - \cos^2(x+c)}{\sin^2(x+c)} = -\csc^2(x+c)$

But $\quad 1 + \cot^2(x+c) = 1 + \frac{\cos^2(x+c)}{\sin^2(x+c)} = \frac{\sin^2(x+c) + \cos^2(x+c)}{\sin^2(x+c)} = \csc^2(x+c)$

Hence $\quad y' = -\csc^2(x+c) = -(1 + \cot^2(x+c)) = -1 - \cot^2(x) = -1 - y^2.$

Thus $f_c(x) = \cot(x+c)$ defines a 1-parameter family of solutions.

Definition (Particular Solution/Singular Solution)
A solution to an ODE which is obtained via specifying the parameters of some parameter family of solutions to the ODE. If a solution cannot be so obtained, we call it a singular solution.

Ex Specify the parameters in any of the parameterized families of solutions above to obtain particular solutions.

Ex A one parameter family of solutions to the ODE $y' = xy^{1/2}$ on $(-\infty, \infty)$ is given by

$$f_c(x) = \left(\tfrac{1}{4}x^2 + c\right)^2, \quad c \geq 0.$$

Indeed

$$f_c'(x) = \left(\left(\tfrac{1}{4}x^2+c\right)^2\right)' = \tfrac{1}{2}x \cdot 2\left(\tfrac{1}{4}x^2+c\right) = x\left(\tfrac{1}{4}x^2+c\right) = x\left(f_c(x)\right)^{1/2}$$

(Note $f_c(x) \geq 0$ since $x^2, c \geq 0$.)

Examples of particular solutions are

$$f_1(x) = \left(\tfrac{1}{4}x^2+1\right)^2, \quad f_0(x) = \tfrac{1}{16}x^4, \quad f_\pi(x) = \left(\tfrac{1}{4}x^2+\pi\right)^2.$$

Now note that $f(x)=0$ also defines a solution:

$$f'(x) = 0 = x \cdot 0 = x \cdot f(x).$$

Yet $f(x)=0$ is not obtainable from specifying c in f_c. Hence $f(x)=0$ is a singular solution.

Δ We end by defining two classes of ODEs. The first class will turn out to be much easier to solve than the second class.

First recall that a polynomial $p(x,y,z)$ is linear if it is of the form

$$p(x,y,z) = ax + by + cz + d, \quad a,b,c,d \in \mathbb{R}.$$

E.g., xy is nonlinear, but $x+z+3$ is. We extend this idea to ODEs.

Definition (Linear ODEs)
If the ODE

$$F(x, y, y', \dots, y^{(n)}) = 0 \quad (\star)$$

may be written in the form

$$\sum_{i=0}^{n} a_i(x) \frac{d^{(i)}y}{dx^{(i)}} = g(x), \quad a_i, g \text{ functions of } x,$$

then $\star$ is said to be a Linear ODE. If this is impossible, $(\star)$ is called nonlinear.

- Why call it linear? Consider the ODE $y'' + 3y + x = 0$. Let $X = y''$, $Y = y$, and consider x as a constant. Then the ODE becomes

$$X + 3Y + x = 0.$$

As a polynomial in the indeterminates X and Y, this is a linear polynomial.

In "English": An ODE $F(x, y, y', \dots, y^{(n)}) = 0$ is linear if it is a linear polynomial in the indeterminates $y, y', \dots, y^{(n)}$.

Ex Which of the following ODEs are linear?

(a) $-cy' + y''' - 6yy' = 0$ (b) $\frac{d^2y}{dx^2} - 3xy = \cos(x)$

(c) $\frac{dy}{dx} - \arctan(y) = 0$ (d) $\frac{dm}{dt} = f_{in}(t) - f_{out}(t)$

(a) This is a nonlinear ODE because of the yy' term.
This ODE is sometimes used to model a fixed wave form traveling in a shallow pool of water. It's called the KdV equation.

(b) This is a linear ODE. It can be placed in the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

where

$$a_2(x) = 1, \quad a_1(x) = 0, \quad a_0(x) = -3x, \quad g(x) = \cos(x).$$

(c) This is a nonlinear ODE because of the $\arctan(y)$ term.

(d) Here, m is the dependent variable and t the independent variable. We observe that this ODE takes the form

$$a_1(t)m' + a_0(t)m = g(t)$$

where $a_1(t) = 1$, $a_0(t) = 0$, $g(t) = f_{in}(t) - f_{out}(t)$.
Hence the ODE is linear.
This ODE may be used to model the amount m of a chemical whose rate of input is f_{in} and rate of output is f_{out} into a tank.

Δ Misc. problem

Ex Find all m so that $y = \cos(mx)$ is a solution to

$$\begin{cases} y'' + 4y = 0 \\ x \in \mathbb{R} \end{cases}$$

We suppose $y = \cos(mx)$ is a solution and check which m work:

$$\begin{aligned} y'' + 4y &= (\cos(mx))'' + 4\cos(mx) \\ &= -m^2\cos(mx) + 4\cos(mx) \\ &= (-m^2 + 4)\cos(mx) \\ &= 0 \end{aligned}$$

This holds for all x only if $-m^2 + 4 = 0$ since $\cos(mx)$ might be nonzero.
Hence

$$m^2 = 4 \iff m = \pm 2.$$