Antiderivatives

Before giving any motivation for what an *antiderivative* is, we shall firstly define as an algebraic operation. Doing it this way will make it easier to understand the connections of antiderivatives with derivatives.

Definition. An **antiderivative** of a function f is a function F such that $F' = f$.

Example 1. Suppose $f(x) = 2x$. Is there a function F we can find such that $F' = 2x$? We see by guessing that $F(x) = x^2$ works: $F'(x) = \frac{dx^2}{dx} = 2x$. However, since the derivative of any constant is zero, we find also that if $G(x) = x^2 + 5$, then $G'(x) = \frac{d(x^2+5)}{dx} = 2x$. Thus, both F and G are antiderivatives of f, and, in fact, there are infinitely many antiderivatives. △

We stress that there are (infinitely) many antiderivatives of a given function.

Theorem 1. *If F is an antiderivative of f, then so is $F + C$ for any real number C.*

Let's see another example.

Example 2. Suppose $f(x) = e^x + \frac{1}{x}$. By guessing, we find that if $F(x) = e^x + \ln|x|$, then $F'(x) = e^x + \frac{1}{x}$. Note that we do need the absolute value in $\ln|x|$ since we are letting x be arbitrary. Again, $G(x) = F(x) + C$ would also be an antiderivative of f. △

The task of this section is going to be to find the "most general" antiderivative of given functions. That is, if F is an antiderivative of f, the "most general" antiderivative of f is $F + C$, where C is an arbitrary constant. With this, we define the **indefinite integral**.

Definition 1. The **indefinite integral** of a function f is the expression $F(x) + C$, where F is an antiderivative of f, and C is an arbitrary constant. We denote the indefinite integral of f by

$$\int f(x)dx = F(x) + C.$$

We note that this is purely notational. The "plus C" term is merely notation to demonstrate that we understand that there are infinitely many antiderivatives of f, whence the name "indefinite integral." One should think of $F(x)+C$ as the collection of *all* antiderivatives of f, whereas F is an antiderivative of f.

We now present properties and examples.

Theorem 2. *The indefinite integral has the following additive and multiplicative properties:*

1. $\int f(x) + g(x)dx = \int f(x)dx + \int g(x)$
2. $\int kf(x)dx = k\int f(x)dx$ *for any real number* k.

Before we can see any examples, we need to know how the indefinite integrals acts on our most basic functions. We first consider simply rational functions.

Theorem 3. *Let $f(x) = x^\alpha$ where $\alpha \neq -1$ is any real number. Then the indefinite integral has the following power rule:*

$$\int f(x)dx = \int x^\alpha dx = \frac{1}{\alpha+1}x^{\alpha+1} + C.$$

The case $\alpha = -1$ is a special case we will handle soon. Another special case is $\alpha = 0$, where

$$\int dx = \int x^0 dx = \frac{1}{0+1}x^{0+1} + C = x + C.$$

Before seeing an example, let's test if this indefinite integral makes sense. We find

$$\frac{d}{dx}(\frac{1}{\alpha+1}x^{\alpha+1} + C) = \frac{d}{dx}\frac{1}{\alpha+1}x^{\alpha+1} + \frac{d}{dx}C = \frac{\alpha+1}{\alpha+1}x^{\alpha+1-1} + 0 = x^\alpha.$$

Thus, $\frac{1}{\alpha+1}x^{\alpha+1} + C$ is in fact an antiderivative for each real number C.

Example 3. *Find the indefinite integral (or most general antiderivative)for* $f(x) = 5x^3 + 3x + 3$.

We use the additive, multiplicative, and power rule properties of the indefinite integral:

$$\begin{aligned}\int 5x^3 + 3x + 3dx &= 5\int x^3dx + 3\int xdx + 3\int dx \\ &= 5\frac{1}{3+1}x^{3+1} + 3\frac{1}{1+1}x^{1+1} + 3x + C \\ &= \frac{5}{4}x^4 + \frac{3}{2}x^2 + 3x + C.\end{aligned}$$

△

Example 4. *Find the indefinite integral (or most general antiderivative)for* $f(x) = 11x^{-2.1} + 3x^{\frac{1}{\pi}}$.

We use the additive, multiplicative, and power rule properties of the indefinite integral:

$$\begin{aligned}\int 11x^{-2.1} + 3x^{\frac{1}{\pi}}dx &= 11\int x^{-2.1}dx + 3\int x^{\frac{1}{\pi}}dx \\ &= 11\frac{1}{-2.1+1}x^{-2.1+1} + 3\frac{1}{1+\frac{1}{\pi}}x^{1+\frac{1}{\pi}} + C \\ &= -\frac{11}{1.1}x^{-1.1} + \frac{3\pi}{1+\pi}x^{1+\frac{1}{\pi}} + C.\end{aligned}$$

△

We now introduce two special cases of indefinite integration. The first is the case $\alpha = 1$ from above.

Theorem 4. *Suppose* $f(x) = x^{-1} = \frac{1}{x}$. *Then*

$$\int f(x)dx = \int x^{-1}dx = \int \frac{1}{x}dx = \ln|x| + C.$$

Note that the absolute values in the indefinite integral is necessary. To justify this theorem, simply differentiate $\ln|x| + C$.

Example 5. *Find the most general antiderivative of* $f(x) = \frac{1+x}{x}$.

The trick here is to recognize that $\frac{1+x}{x} = \frac{1}{x} + \frac{x}{x} = \frac{1}{x} + 1$. Thus

$$\begin{aligned}\int f(x)dx &= \int \frac{1}{x} + 1dx \\ &= \int \frac{1}{x}dx + \int dx \\ &= \ln|x| + x + C.\end{aligned}$$

△

Theorem 5. *Suppose* $f(x) = e^x$. *Then*

$$\int e^x dx = e^x + C.$$

Example 6. *Find the most general antiderivative of* $f'(x) = e^x + 3x$.

This is just asking us to integrate $f'(x) = e^x + 3x$:

$$\begin{aligned}\int f'(x)dx &= \int e^x + 3xdx \\ &= \int e^x dx + 3\int xdx \\ &= e^x + 3\frac{1}{1+1}x^{1+1} + C \\ &= e^x + \frac{3}{2}x^2 + C.\end{aligned}$$

△

In closing: What you might have noticed is that indefinite integration in a sense undoes differentiation. In fact, we have the following theorem.

Theorem 6. *Suppose f is differentiable. Then*

$$\int f'(x)dx = f(x) + C.$$

Substitution

In this section we give a method for integrating more difficult functions. The method given here is a sort of reversal of the chain rule.

Theorem 7. *Suppose f and g are functions such that $f(g(x))$ makes sense. Then*

$$\int f'(g(x))g'(x)dx = f(g(x)) + C.$$

Let's justify this theorem. We make the substitution $g(x) = u$. Then, by differentiating with respect to x, we get

$$g'(x) = \frac{du}{dx}$$

and so

$$g'(x)dx = du.$$

Note that this treatment of dx and du is barely mathematically sound, but it will do for now.

While this theorem is nice, it is not quite useful because the integrand rarely fits the patter of $f'(g(x))g'(x)$ perfectly. However, the algebra involved in the substitution done above will usually work and this is what we are to stress. We do this by plenty of examples. The main theme is that we guess what our u should be in the substitution so as to make the integral simpler. The mains steps are

1. Test a substitution with u.
2. Solve for dx.
3. Make the u-substitution and substitute for dx.
4. Evaluate the integral in terms of u.
5. Substitute back to obtain an answer in terms of x.

Example 7. *Find the indefinite integral of $f(x) = (x+1)^{20}$*

First notice that we could foil $(x+1)^{20}$ and then integrate if we wished. But this would take forever, so let's try doing the substitution. Here we take $u = x+1$ and so $du = \frac{d}{dx}(x+1)dx = dx$ and $(x+1)^{20} = u^{20}$.

Thus, we find

$$\int (x+1)^{20}dx = \int u^{20}du = \frac{1}{21}u^{21} + C = \frac{1}{21}(x+1)^{21}.$$

△

Example 8. *Find the indefinite integral of $f(x) = \frac{4}{(2x+4)^4}$.*

We wish to evaluate the indefinite integral

$$\int \frac{4}{(2x+4)^4}dx.$$

We try $u = 2x+4$. From this,

$$du = \frac{d}{dx}(2x+4)dx = 2dx$$

and so

$$dx = \frac{du}{2}.$$

Then,

$$\int \frac{4}{(2x+4)^4}dx = \int \frac{4}{u^4}\frac{du}{2} = 2\int u^{-4}du = -\frac{2}{3}u^{-3} + C = -\frac{2}{3}(2x+3)^{-3} + C.$$

△

Example 9. $\frac{x}{x^2+1}$

$x^3 e^{x^4}$

$(x+1)(2x+x^2+1)^{50}$

$\triangle$

The Definite Integral

In this section, we define what is called the definite integral of a function. In the examples, we will only consider nonnegative functions for this section so that we get a nice geometric idea of what an indefinite integral is. However, before we give such a characterization, we define the definite integral using somewhat a abstract tool called a Riemann sum. So let us construct what we call a definite integral.

Suppose f is a continuous function on an interval $[a, b]$ and let $\Delta x = \frac{b-a}{n}$ for some positive integer n. Partition the interval $[a, b]$ into n even subintervals of length Δx. Label the endpoints of these subintervals $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. We define

$$\begin{aligned}
\text{right hand sum} &= f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \\
&= \sum_{k=1}^{n} f(x_k)\Delta x \\
\text{left hand sum} &= f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x \\
&= \sum_{k=0}^{n-1} f(x_k)\Delta x.
\end{aligned}$$

We are ready to define the **definite integral** of f. Since f is continuous on $[a, b]$, we can show that

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\Delta x.$$

We call their common value the definite integral of f form a to b and write it as

$$\int_a^b f(x)dx.$$

If $f \geq 0$, then $\int_a^b f(x)dx$ is the area of the region under the graph of f on $[a, b]$.

Example 10. *Approximate* $\int_0^5 x^2 + 1dx$ *by the left and right sums when* $n = 5$.

Let's do this is steps.

1. Partition the interval $[0, 5]$ into 5 equal subintervals of length $\Delta x = \frac{5-0}{5} = 1$. Thus, we obtain the endpoints $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5$.

2. Compute the left hand sum.

$$\begin{aligned}
\sum_{k=0}^{4} f(x_k)\Delta x &= f(0) + f(1) + f(2) + f(3) + f(4) \\
&= (0 + 1) + (1 + 1) + (4 + 1) + (9 + 1) + (16 + 1) \\
&= 35
\end{aligned}$$

3. Similarly compute the left hand sum.

△

Example 11. *Compute* $\int_0^5 x + 1dx$ *by finding the area of an appropriate geometric region.*

Here, you must draw the graph of f over $[0, 5]$ and then find the appropriate area under the curve. △