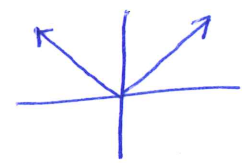
PRACTICE EXAM Solutions

PART A

1\. (a) True

(An example : $f(x) = |x|$ on $(-\infty, \infty)$.

f is continuous everywhere but not differentiable at $x = 0$



(b) True

By the extreme value theorem

(Every absolute extrema is also a relative extrema).

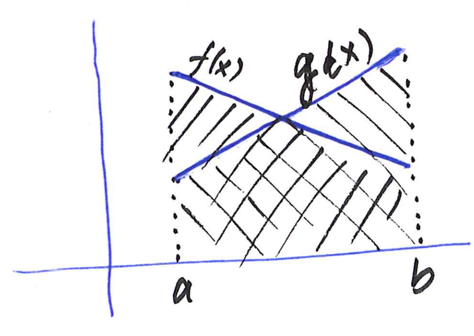
(c) True

By the definition of a critical value.

(d). False

(Counterexample :

Areas under the graphs of $f(x)$ and $g(x)$ are the same but $f(x) \neq g(x)$)



(e). <u>False</u>.

(Counterexample : Let $f(x) = x^{4/3}$.

Then, $f(x)$ is differentiable everywhere because $f'(x) = \frac{4}{3} x^{1/3}$ which exists everywhere.

But $f''(x) = \frac{4}{9} x^{-2/3}$ which does not exist at $x=0$, so $f'(x)$ is not differentiable at $x=0$).

(f). <u>True</u>.

<u>PART B</u> 2. (a)

$$\ln(x) = \frac{1}{2}\ln(9) + \ln(5) - \ln(6).$$

$$\ln(x) = \ln(9)^{\frac{1}{2}} + \ln(5) - \ln(6)$$

$$\ln(x) = \ln(3) + \ln(5) - \ln(6)$$

$$\ln(x) = \ln(3\cdot 5) - \ln(6)$$

$$\ln(x) = \ln\left(\frac{3\cdot 5}{6}\right)$$

$$e^{\ln(x)} = e^{\ln\left(\frac{15}{6}\right)}.$$

So, $x = \frac{15}{6}$ which lies in the domain of $\ln(x)$.

2.(b). $\log_3(9x+12) = 1 + \log_3(x+3)$

$\Rightarrow \log_3(9x+12) = \log_3(3) + \log_3(x+3)$

$\Rightarrow \log_3(9x+12) = \log_3[3.(x+3)]$

$\Rightarrow \log_3(9x+12) = \log_3(3x+9).$

$\Rightarrow 3^{\log_3(9x+12)} = 3^{\log_3(3x+9)}$

$\Rightarrow 9x+12 = 3x+9$

$\Rightarrow 6x = -3$

or $x = -\frac{1}{2}$ which lies in the domains

of $\log_3(9x+12)$ and $\log_3(x+3)$.

(c). $\log_2(16x-14) + \log_2(x+8) = 2\log_2(4x).$

$\Rightarrow \log_2[(16x-14).(x+8)] = \log_2(4x)^2.$

$\Rightarrow 2^{\log_2[(16x-14)\cdot(x+8)]} = 2^{\log_2(4x)^2}$

$\Rightarrow (16x-14)(x+8) = (4x)^2.$

$\Rightarrow 16x^2 + 114x - 112 = 16x^2$

$\Rightarrow 114x = 112$

or $x = 112/114$ which is in the domains of

$\log_2(16x-4)$, $\log_2(x+8)$ and $\log_2(4x)$.

2. (d). $\log_7(x) = \frac{1}{3}\log_7(8) + \log_7(7) - \log_7(2)$.

$$\Rightarrow \log_7(x) = \log_7(8)^{\frac{1}{3}} + \log_7 7 - \log_7(2)$$

$$\Rightarrow \log_7(x) = \log_7(2) + \log_7(7) - \log_7(2)$$

$$\Rightarrow \log_7(x) = \log_7\left(\frac{2\cdot 7}{2}\right)$$ (No need for this step)

$$\Rightarrow \log_7(x) = \log_7(7)$$

$$\Rightarrow 7^{\log_7(x)} = 7^{\log_7(7)}$$

$\Rightarrow \quad x = 7$ which lies in the domain of $\log_7(x)$.

2. (e). $\log_4(10x+15) = 1 + \log_4(x+5)$

$$\Rightarrow \log_4(10x+15) = \log_4(4) + \log_4(x+5)$$

$$\Rightarrow \log_4(10x+15) = \log_4[4\cdot(x+5)]$$

$$\Rightarrow 4^{\log_4(10x+15)} = 4^{\log_4(4x+20)}$$

$$\Rightarrow 10x+15 = 4x+20$$

$$\Rightarrow 6x = 5$$

or $x = 5/6$ which is in the domains of $\log_4(10x+15)$ and $\log_4(x+5)$.

Part B

3(a) Let $x = \log_7\left(\frac{1}{7^7}\right)$

Then, $x = \log_7(7^{-7})$

$\Rightarrow x = -7 \cdot \log_7(7)$

$\Rightarrow x = -7 \cdot 1$

$\Rightarrow x = -7.$

3(b). Let $x = 2^{2\log_2(2)}$

Then, $x = 2^{\log_2 (2)^2}$

$\Rightarrow x = 2^2 = 4$

3(c). Let $x = \log_5(e^{\ln(5)})$

Then, $x = \dfrac{\log_e e^{\ln(5)}}{\log_e(5)}$

$= \dfrac{\ln(5)}{\log_e(5)}$

$= \dfrac{\ln(5)}{\ln(5)} = 1.$

3.(d) Let $x = \log_{12}(3) + \log_{12}(6) + \log_{12}(8)$.

Then,

$$
\begin{aligned}
x &= \log_{12}(3 \cdot 6 \cdot 8) \\
&= \log_{12}(144) \\
&= \log_{12}(12)^2 \\
&= 2 \log_{12}(12) \\
&= 2 \cdot 1 \\
&= 2 .
\end{aligned}
$$

PART C

4(a) $\lim_{x \to 2} \frac{x^2 - x + 6}{x - 2}$

$$\lim_{x \to 2^+} \frac{x^2 - x + 6}{x - 2} = +\infty$$

and

$$\lim_{x \to 2^-} \frac{x^2 - x + 6}{x - 2} = -\infty .$$

Since LSL $\neq$ RSL,

$\lim_{x \to 2} \frac{x^2 - x + 6}{x - 2}$ DNE.

4(b) $\lim_{h \to 0} \frac{(1+h^2) - 1}{h}$

$$= \lim_{h \to 0} \frac{(1+h)(1+h) - 1}{h}$$

$$= \lim_{h \to 0} \frac{(1 + 2h + h^2) - 1}{h}$$

$$= \lim_{h \to 0} \frac{2h + h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2+h)}{h}$$

$$= \lim_{h \to 0} 2 + h = 2 .$$

(c). $\lim\limits_{x \to -5} \dfrac{x^2+6x+5}{x^2+4x-5}$

$= \lim\limits_{x \to -5} \dfrac{(x+5)(x+1)}{(x+5)(x-1)}$

$= \lim\limits_{x \to -5} \dfrac{x+1}{x-1}$

$= \dfrac{\lim\limits_{x \to -5} (x+1)}{\lim\limits_{x \to -5} (x-1)} = \dfrac{-5+1}{-5-1} = \dfrac{-4}{-6} = \dfrac{2}{3}.$

(d). $\lim\limits_{x \to 2} f(x)$ where $f(x) = \begin{cases} 8-x^2 & , x \le 2 \\ x-3 & , x > 2 \end{cases}$

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (8-x^2) = 4$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (x-3) = -1$

Since LSL $\neq$ RSL,

$\lim\limits_{x \to 2} f(x)$ DNE.

5. (a). $f(x) = \dfrac{3+9e^{-5x}}{12+5e^{-5x}}$.

$$\lim_{x\to\infty} \frac{3+9e^{-5x}}{12+5e^{-5x}} = \lim_{x\to\infty} \frac{3}{12} = \frac{1}{4}$$

$$\lim_{x\to-\infty} \frac{3+9e^{-5x}}{12+5e^{-5x}} = \lim_{x\to-\infty} \frac{9e^{-5x}}{5e^{-5x}}$$

$$= \lim_{x\to-\infty} \frac{9}{5} = \frac{9}{5}$$

So, $y = \frac{1}{4}$ and $y = \frac{9}{5}$ are the Horizontal Asymptotes of $f(x)$.

(b). $f(x) = \dfrac{5}{2+e^{-4x}}$

$$\lim_{x\to\infty} \frac{5}{2+e^{-4x}} = \frac{5}{2}$$

$$\lim_{x\to-\infty} \frac{5}{2+e^{-4x}} = 0$$

So, $y = \frac{5}{2}$ and $y = 0$ are the Horizontal asymptotes.

5(c). $f(x) = \frac{x^3 + 3x}{6 - 4x^3}$.

$$\lim_{x \to \infty} \frac{x^3 + 3x}{6 - 4x^3} = \frac{1}{-4} = -\frac{1}{4}.$$

$$\lim_{x \to -\infty} \frac{x^3 + 3x}{6 - 4x^3} = \frac{-1}{4}.$$

So, $y = -\frac{1}{4}$ is the horizontal asymptote.

PART D

6. $f(x) = \frac{3x^2 + 4x^2}{(2x^2 + 4x)^3} = \frac{7x^2}{(2x^2 + 4x)^3}$

Let $g(x) = 7x^2$ and $h(x) = (2x^2 + 4x)^3$.

Then, by quotient rule,

$$f'(x) = \frac{g'(x)\,h(x) - g(x)\,h'(x)}{(h(x))^2}.$$

Now,

$$g'(x) = \frac{d}{dx}(7x^2) = 7\frac{d}{dx}(x^2) = 7 \cdot 2x = 14x$$

and

$$h'(x) = \frac{d}{dx}\left[(2x^2 + 4x)^3\right] = 3(2x^2 + 4x)^2 \cdot (4x + 4)$$

So,

$$f'(x) = \frac{(14x)(2x^2 + 4x)^3 - 7x^2 \cdot \left[3(2x^2 + 4x)^2 \cdot (4x + 4)\right]}{(2x^2 + 4x)^6}.$$

7. $f(x) = 3x e^x$.

General equation of the tangent line :

$$(y - y_1) = m_{tan} (x - x_1).$$

m_{tan} at $x = 0$ is $f'(0)$.

$$f'(x) = 3 (1 \cdot e^x + x \cdot e^x) \quad \text{(Product Rule)}.$$

$$f'(0) = 3 (1 \cdot e^0 + 0 \cdot e^0)$$
$$= 3$$

Also, $(x_1, y_1) = (0, f(0)) = (0, 3 \cdot 0 \cdot e^0) = (0,0)$

Thus, the tangent line is

$$(y - 0) = 3(x - 0)$$

or $y = 3x$.

8. $f(x) = x^2 e^{-x}$.

(a) $f'(x) = 2x\, e^{-x} + x^2 \cdot e^{-x} \cdot (-1)$.

$= e^{-x}(2x - x^2)$.

- $f'(x) = 0$

when $2x - x^2 = 0$

or $x = 0$, $x = 2$.

- $f'(x)$ exists everywhere in the domain.

So, Critical values : $x = 0$, $x = 2$.

(b).

x		0		2	
test points	-1		0.5		3
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$		$\nearrow$		$\searrow$

$f(x)$ increasing on $(0, 2)$

$f(x)$ decreasing on $(-\infty, 0) \cup (2, \infty)$.

(c). Relative maxima at $(2, f(2)) = (2, 4e^{-2})$

Relative minima at $(0, f(0)) = (0, 0)$

9. $f(x) = \dfrac{-x^2}{x+2}$

(a). $f'(x) = \dfrac{-2x \cdot (x+2) - (-x)^2 \cdot 1}{(x+2)^2}$

$= \dfrac{-2x^2 - 4x + x^2}{(x+2)^2} = \dfrac{-x^2 - 4x}{(x+2)^2}$

$f'(x) = 0$ when $-x^2 - 4x = 0$

or $x = 0$, $x = -4$

So, critical points of $f(x)$ are $x = 0$ and $x = -4$

(Note that $x = -2$ does not belong to the domain of $f(x)$, so $f'(x)$ exists everywhere in the domain).

(b).

x		-4		0	
test points	-5		-1		1
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$		$\nearrow$		$\searrow$

So,

$f(x)$ is increasing on $(-4,-2) \cup (-2,0)$

$f(x)$ is decreasing on $(-\infty,-4) \cup (0,\infty)$

(c). Relative maxima at $(0, f(0)) = (0, 0)$

Relative minima at $(-4, f(-4)) = (-4, 8)$.

10. $f(x) = 2\sqrt[3]{x^5} - 5\sqrt[3]{x^2}$.

(a). $f(x) = 2x^{\frac{5}{3}} - 5x^{\frac{2}{3}}$

$f'(x) = 2 \cdot \frac{5}{3} x^{2/3} - 5 \cdot \frac{2}{3} x^{-\frac{1}{3}}$.

- $f'(x) = 0$ when $\frac{10}{3} x^{2/3} = \frac{10}{3} x^{-1/3}$

or $\frac{x^{2/3}}{x^{-1/3}} = 1$

or $x^{\frac{2}{3}+\frac{1}{3}} = 1$

or $x = 1$

- $f'(x)$ does not exist at $x=0$

So, Critical points : $x = 0$, $x = 1$.

(b).

x		0		1	
test points	-1		0.5		2
$f'(x)$	$+$	DNE	$-$	0	$+$
$f(x)$	$\nearrow$		$\searrow$		$\nearrow$

So, f is increasing on $(-\infty, 0) \cup (1, \infty)$
f is decreasing on $(0, 1)$.

(c) Relative maxima at $(0, f(0)) = (0, 0)$
Relative minima at $(1, f(1)) = (1, 0)$

11. $$g''(x) = \frac{(x+4)^2(x-3)}{(x-5)^7}.$$

(a) $(x+4)^2$ is always positive, so the sign of $g''(x)$ depends on the signs of $(x-3)$ and $(x-5)^7$.

		3		5	
$(x-3)$	$-$		$+$		$+$
$(x-5)^7$	$-$		$-$		$+$
$g''(x)$	$+$	0	$-$	DNE	$+$

So, g is concave up on $(-\infty,3)\cup(5,\infty)$ and concave down on $(3,5)$.

(b) Since the concavity changes at $(3, f(3))$, and $g''(3)=0$

$x=3$ is the x-coordinate of the inflection point.

12. $f(x) = 2x^4 + 52x^3 - 14x + 13$

(a) $f'(x) = 8x^3 + 156x^2 - 14$

$f''(x) = 24x^2 + 312x$

$f''(x) = 0$ when $x = 0$ and $x = -13$

$f''(x)$ exists everywhere.

x		-13		0	
test points	-14		-1		1
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\cup$		$\cap$		$\cup$

So, f is concave up on $(-\infty, -13) \cup (0, \infty)$

f is concave down on $(-13, 0)$.

(b). Inflection points : $(-13, f(-13)) = (-13, 171561)$

and $(0, f(0)) = (0, 13)$

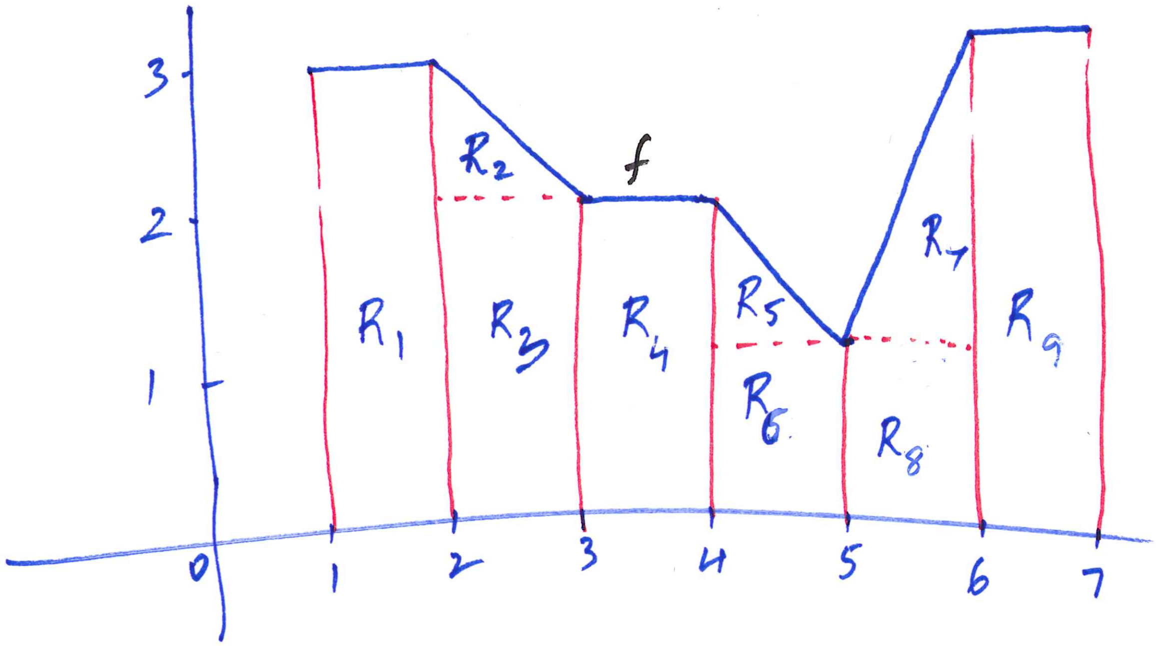
because $f''(-13) = 0$, $f''(0) = 0$ and

the graph of f changes concavity at $(-13, f(-13))$

and $(0, f(0))$.

PART E

13. Find $\int_1^7 f(x)\, dx$



$\int_1^7 f(x)$ = Area of the region below the graph of f and above the x-axis.

= Sum of areas of regions R_1, R_2, R_3, R_4, R_5, R_6, R_7 and R_8

$$= (3\cdot 1) + \left(\tfrac{1}{2}\cdot 1\cdot 1\right) + (2\cdot 1) + (2\cdot 1) + \left(\tfrac{1}{2}\cdot 1\cdot 1\right) + (1\cdot 1) + \left(\tfrac{1}{2}\cdot 2\cdot 1\right) + (1\cdot 1) + (3\cdot 1)$$

$$= 3 + \tfrac{1}{2} + 2 + 2 + \tfrac{1}{2} + 1 + 1 + 1 + 3 = 14$$

14. $f(x) = x^4 + e^x + \frac{2}{x}$, we need to find $\int_1^2 f(x)\,dx$.

An antiderivative of f is $F(x) = \frac{x^5}{5} + e^x + 2\ln|x|$.

So, by Fundamental theorem of Calculus,

$$\int_1^2 f(x)\,dx = F(2) - F(1)$$

$$= \left[\frac{2^5}{5} + e^2 + 2\ln(2)\right] - \left[\frac{1^5}{5} + e^1 + 2\ln(1)\right]$$

$$= \left(\frac{32}{5} + e^2 + 2\ln(2)\right) - \left(\frac{1}{5} + e\right)$$

$$= \frac{31}{5} + e^2 - e + 2\ln(2).$$

Part F

15. $f(x) = e^{-x^2}$.

(a) Domain : $(-\infty, \infty)$

(b) x- intercept DNE.
y-intercept : $(0, 1)$.

(c). $\lim_{x \to \infty} e^{-x^2} = 0$

$\lim_{x \to -\infty} e^{-x^2} = 0$

So, $y = 0$ is the Horizontal asymptote.

(d). $f'(x) = e^{-x^2} \cdot (-2x)$

$f'(x) = 0$ when $x = 0$
and $f'(x)$ exists everywhere.
So, critical point is $x = 0$.

x		0	
Test points	-1		1
$f'(x)$	$+$	0	$-$
$f(x)$	↗		↘

so, f increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

(e). $f''(x) = -2x\, e^{-x^2} \cdot (-2x) + e^{-x^2} \cdot (-2)$

$f''(x) = 0$ when $4x^2 e^{-x^2} - 2e^{-x^2} = 0$

or $x = \pm \frac{1}{\sqrt{2}}$.

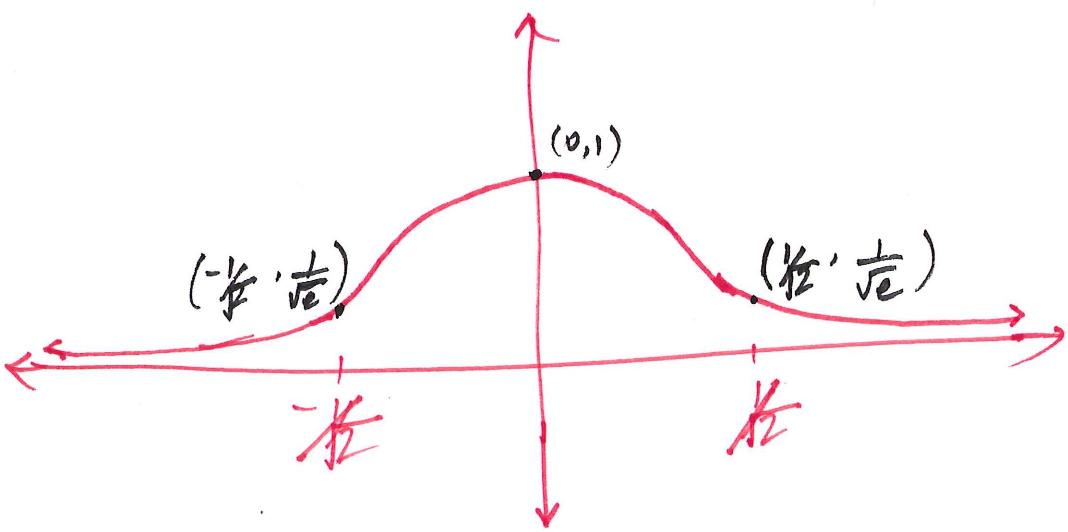
and $f''(x)$ exists everywhere.

x		$-\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$	
Test points	-1		0		1
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\cup$		$\cap$		$\cup$

So, f is concave up on $(-\infty, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, \infty)$

and concave down on $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

(g)



(4). $f(x)$ attains an absolute maximum at $x=0$ because f increases on $(-\infty, 0)$ and decreases on $(0, \infty)$ as can be seen from the graph.

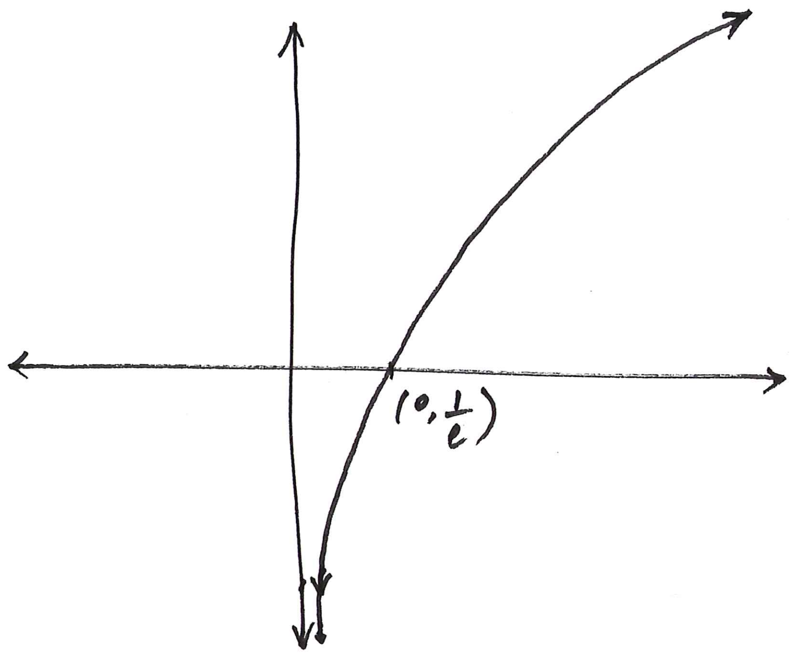
Absolute minimum DNE.

(e.) $f''(x) = -\frac{1}{x^2} < 0$ on $(0, \infty)$

So, f is concave down on the entire domain.

f is never concave up.

(g).



(h). $f(1) = 1 + \ln(1) = 1.$

$(1, 1)$ is the absolute minimum as f increases on $[1, \infty)$

Absolute maximum DNE as $\lim_{x \to \infty} f(x) = \infty.$