

1

Solutions Practice Exam 2

① $f(x+h) = \frac{3}{2(x+h)-9}$ $f(x) = \frac{3}{2x-9}$

↓ plug ← plug

$$f'(x) \stackrel{\text{def}}{=} \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{3}{2(x+h)-9} - \frac{3}{2x-9}}{h} =$$

$$= \lim_{h \rightarrow 0} \frac{\left(\frac{1}{2x+2h-9} - \frac{1}{2x-9} \right)}{h} = 3 \lim_{h \rightarrow 0} \frac{(2x-9) - (2x+2h-9)}{(2x+2h-9)(2x-9) \cdot \frac{1}{h}} =$$

$$= 3 \lim_{h \rightarrow 0} \frac{-2h}{(2x+2h-9)(2x-9)} \cdot \frac{1}{h} = \frac{-6}{(2x+2 \cdot 0 - 9)(2x-9)} = \boxed{-\frac{6}{(2x-9)^2}}$$

② $f(x+h) = 2(x+h)^2 - (x+h)$ $f(x) = 2x^2 - x$

↓ plug ← plug

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[2(x+h)^2 - (x+h)] - [2x^2 - x]}{h} =$$

$$= \lim_{h \rightarrow 0} \frac{[2x^2 + 2h^2 + 4xh - x - h] - [2x^2 - x]}{h} = \lim_{h \rightarrow 0} \frac{h(2h + 4x - 1)}{h} =$$

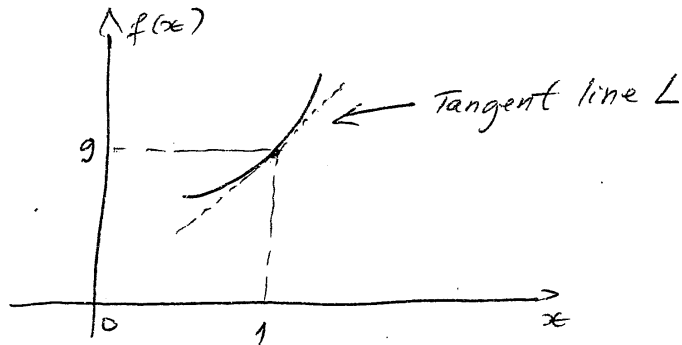
$$= 2 \cdot 0 + 4x - 1 = \boxed{4x - 1}$$

③ The x -values where f is not differentiable are:

- $x = -3$ (vertical tangent)
- $x = 1$ (discontinuity)
- $x = 2$ (corner)

(2)

(4)



$$f(x) = x^9 + 8x^2 \Rightarrow f(1) = \boxed{9}$$

$$f'(x) = 9x^8 + 16x \Rightarrow f'(1) = \boxed{25} \text{ (slope of } L)$$

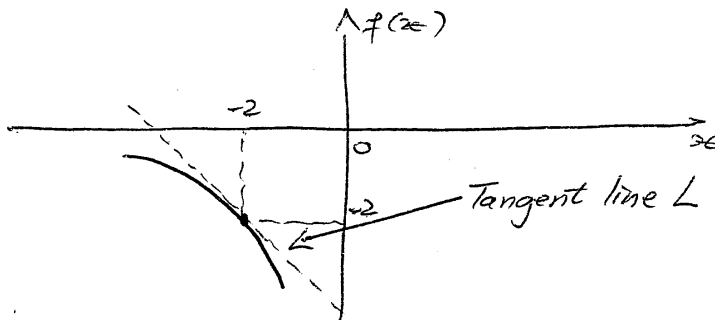
Equation of the line L of slope 25 and passing through $(1, 9)$:

$$\boxed{y - 9 = 25(x - 1)}$$

You can either leave it like this or bring it to its slope-intercept form:

$$\boxed{y = 25x - 16}$$

(5) Same procedure:



$$f(x) = \frac{4}{x} \Rightarrow f(-2) = \boxed{-2}$$

$$f'(x) = 4 \cdot \left(-\frac{1}{x^2}\right) = -\frac{4}{x^2} \Rightarrow f'(-2) = \boxed{-1} \text{ (slope of } L)$$

3

Equation of the line L of slope -1 and passing through $(-2, -2)$:

$$y + 2 = -1(x + 2)$$

Or, in slope-intercept form: $y = -x - 4$

6

$$a) f'(x) = 6x^2 + 35x^6 + 11 \cdot 13 x^{12} = 6x^2 + 35x^6 + 143x^{12}$$

$$b) g'(x) = 3 \cdot \frac{1}{x} + \frac{2}{3}x$$

$$c) f'(x) = 1 + 4x + 9x^2$$

$$d) h'(x) = \frac{3}{2}x^{\frac{3}{2}-1} + e^x = \frac{3}{2}x^{\frac{1}{2}} + e^x = \frac{3}{2}\sqrt{x} + e^x$$

$$e) f'(x) = (x^2+1)'e^x + (x^2+1)(e^x)' = 2x \cdot e^x + (x^2+1) \cdot e^x$$

↑
product rule

$$= e^x(x^2 + 2x + 1) = e^x(x+1)^2$$

$$f) h'(x) = (\sqrt{x})' \cdot \ln x + \sqrt{x} \cdot (\ln x)' = \frac{1}{2\sqrt{x}} \cdot \ln x + \sqrt{x} \cdot \frac{1}{x} =$$

↑
product rule

$$= \frac{1}{2\sqrt{x}} \cdot \ln x + \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{x}} \left(\frac{\ln x}{2} + 1 \right)$$

$$g) g'(x) = \frac{x' \cdot (\ln x) - x \cdot (\ln x)'}{(\ln x)^2} = \frac{1 \cdot \ln x - x \cdot \frac{1}{x}}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$$

↑
quotient rule

4

$$h) h'(x) = (e^x \ln x)' + 5' = (e^x \cdot \ln x)' = (e^x)' \cdot \ln x + e^x \cdot (\ln x)' =$$

↑
sum
rule
↑
product
rule

$$= e^x \cdot \ln x + e^x \cdot \frac{1}{x} = \boxed{e^x \left(\ln x + \frac{1}{x} \right)}$$

$$i) g'(x) = \frac{(2x^2)' (x^3 + x^4)^2 + x^2 [(x^3 + x^4)^2]'}{(x^3 + x^4)^4} =$$

↑
quotient
rule

due to chain rule

$$= \frac{2x \cdot (x^3 + x^4)^2 + x^2 \cdot 2(x^3 + x^4) \cdot (x^3 + x^4)'}{(x^3 + x^4)^4} =$$

$$= 2x (x^3 + x^4) \cdot \frac{[(x^3 + x^4) + x(3x^2 + 4x^3)]}{(x^3 + x^4)^3} =$$

$$= 2x \cdot \frac{5x^4 + 4x^3}{(x^3 + x^4)^3} = \boxed{\frac{2x^4(5x + 4)}{(x^3 + x^4)^3}}$$

$$j) h'(x) = (x+2)' \ln x + (x+2)(\ln x)' = 1 \cdot \ln x + (x+2) \cdot \frac{1}{x} =$$

↑
product
rule

$$= \boxed{\ln x + \frac{x+2}{x}}$$

$$k) f'(x) = 2016 \cdot (x+1)^{2015} \cdot \underbrace{(x+1)'}_1 = \boxed{2016 \cdot (x+1)^{2015}}$$

↑
chain rule

5

$$l) h'(x) = 5 (e^x + x)^4 \cdot (e^x + x)' = \boxed{5 (e^x + x)^4 \cdot (e^x + 1)}$$

↑
chain rule

$$m) g'(x) = 11 (5x + e^x)^{10} \cdot (5x + e^x)' = \boxed{11 (5x + e^x)^{10} \cdot (5 + e^x)}$$

↑
chain rule

$$h) f'(x) = 5 (x^3 + 2x - 8)^4 \cdot (x^3 + 2x - 8)' = \boxed{5 (x^3 + 2x - 8)^4 (3x^2 + 2)}$$

↑
chain rule

$$o) g'(x) = - \frac{1}{\int (x^3 + 4x)^4 \int^2} \cdot \int (x^3 + 4x)^4 \int' =$$

↑
chain rule

again due to chain rule

$$= - \frac{1}{(x^3 + 4x)^{85}} \cdot 4 (x^3 + 4x)^3 \cdot (x^3 + 4x)' =$$

$$= - \frac{1}{(x^3 + 4x)^5} \cdot 4 (3x^2 + 4) = \boxed{- \frac{4(3x^2 + 4)}{(x^3 + 4x)^5}}$$

$$p) f'(x) = 23 (x^2 + \frac{x^3}{3})^{22} \cdot (x^2 + \frac{x^3}{3})' = 23 (x^2 + \frac{x^3}{3}) (2x + x^2) =$$

↑
chain rule

$$= 23 x^3 (1 + \frac{x}{3}) (2 + x) = \boxed{23 x^3 (\frac{x^2}{3} + \frac{5x}{3} + 2)}$$

6

$$q) g'(x) = 16 (\ln x + 20)^{15} \cdot (\ln x + 20)' = 16 (\ln x + 20)^{15} \cdot \frac{1}{x}$$

↑
chain
rule

$$r) f'(x) = e^{7x^2-5} \cdot (7x^2-5)' = e^{7x^2-5} \cdot 14x = 14x e^{7x^2-5}$$

↑
chain
rule

$$s) g'(x) = 3^{4x^3+3} \cdot (4x^3+3)' \cdot \ln 3 = 3^{4x^3+3} \cdot (12x^2) \cdot \ln 3 =$$

↑
chain
rule

$$= 12 (\ln 3) \cdot x^2 \cdot 3^{4x^3+3}$$

$$t) h'(x) = 5^{x^2+10} \cdot (x^2+10)' \cdot \ln 5 = 5^{x^2+10} \cdot (2x) \cdot \ln 5 =$$

↑
chain
rule

$$= 2 \cdot (\ln 5) \cdot x \cdot 5^{x^2+10}$$

$$u) f'(x) = \frac{1}{3x^2+x} \cdot (3x^2+x)' = \frac{6x+1}{3x^2+x}$$

↑
chain
rule

7

$$v) f'(x) = \frac{1}{2x^4 - x} \cdot (2x^4 - x)' \cdot \frac{1}{\ln 3} = \boxed{\frac{1}{\ln 3} \cdot \frac{8x^3 - 1}{2x^4 - x}}$$

↑
chain
rule

$$w) f'(x) = \frac{1}{x^{-1}} \cdot (x^{-1})' \cdot \frac{1}{\ln 10} = \frac{1}{\frac{1}{x}} \cdot (-1 \cdot x^{-2}) \cdot \frac{1}{\ln 10} =$$

↑
chain
rule

$$= x \cdot \left(-\frac{1}{x^2}\right) \cdot \frac{1}{\ln 10} = \boxed{-\frac{1}{(\ln 10) \cdot x}}$$

7) Domain $f = (-\infty, 0) \cup (0, \infty)$.

a) With quotient rule, f is differentiable at every point of its domain, hence the only critical points can occur where $f'(x) = 0$.

$$f'(x) = \frac{(x+5)'x^2 - (x+5) \cdot (x^2)'}{(x^2)^2} = \frac{1 \cdot x^2 - (x+5) \cdot 2x}{x^4} = \boxed{\frac{-x-10}{x^3}}$$

↑
quotient
rule

$$f'(x) = 0 \iff -x - 10 = 0 \iff \boxed{x = -10}$$

↑
the only critical value for f

b) We have:

x	$-\infty$	-10	0	$+\infty$
$-x-10$	+	0	-	-
x^3	-	-	0	+
$f'(x)$	-	0	+	-

8

f is increasing on the interval(s) on which $f' > 0$, hence (from the table) $\boxed{\text{on } (-10, 0)}$.

c) f is decreasing on the interval(s) on which $f' < 0$, hence (from the table) $\boxed{\text{on } (-\infty, -10) \cup (0, \infty)}$.

8

a) g' is well defined for every $x \in \mathbb{R}$, hence the only critical values for g can occur where $g'(x) = 0$.

$$g'(x) = 0 \Leftrightarrow x^2(x^2 - 1) = 0 \Leftrightarrow \boxed{x \in \{-1, 0, 1\}}$$

So these are the critical values for g .

b) We have:

x	$-\infty$	-1	0	1	$+\infty$	
x^2		+	+	+	+	
$x^2 - 1$		+	0	-	0	+
$g'(x) = x^2(x^2 - 1)$		+	0	-	0	+
$g(x)$		\nearrow	\searrow	\searrow	\nearrow	

g is increasing on the interval(s) on which $g' > 0$, hence (from the table) $\boxed{\text{on } (-\infty, -1) \cup (1, \infty)}$.

c) g is decreasing on the interval(s) on which $g' < 0$, hence (from the table) $\boxed{\text{on } (-1, 0) \cup (0, 1)}$.

9

$$\textcircled{9} \quad f'(x) = 4x^3 + 6x^2 - 24x = 2(2x^3 + 3x^2 - 12x)$$

$$f''(x) = 2(6x^2 + 6x - 12) = 12(x^2 + x - 2) = 12(x+2)(x-1)$$

a) We have:

x	$-\infty$		-2		1		$+\infty$
$f''(x) = 12(x+2)(x-1)$		+	0	-	0	+	
$f(x)$		∪	$f(-2)$ inflection	∩	$f(1)$ inflection	∪	

f is concave up where $f'' > 0$, hence on $(-\infty, -2) \cup (1, \infty)$

f is concave down where $f'' < 0$, hence on $(-2, 1)$.

b) Inflection points: at $x = -2$ and at $x = 1$, because these are the values where $f''(x)$ becomes equal to 0 and changes sign.

$$\textcircled{10} \quad g''(x) = \frac{(x+2)(x+5)}{(x-4)^2}$$

a) We have:

x	$-\infty$		-5		-2		4		$+\infty$
$(x+2)(x+5)$		+	0	-	0	+		+	
$(x-4)^2$		+		+		+	0		+
$g''(x)$		+	0	-	0	+			+
$g(x)$		∪	$g(-5)$ inflection	∩	$g(-2)$ inflection	∪			∪

10

g is concave up where $g'' > 0$, hence on $(-\infty, -5) \cup (-2, 4) \cup (4, \infty)$.

g is concave down where $g'' < 0$, hence on $(-5, -2)$.

b) Inflection points: $\boxed{\text{at } x = -5 \text{ and at } x = -2}$, because these are the values of x where $g''(x)$ becomes equal to 0 and changes sign.

11

$$a) \lim_{x \rightarrow +\infty} \frac{16x^2 + 3x^2 + 6}{5x^3 + 6x^2 + 2} = \lim_{x \rightarrow +\infty} \frac{19x^2 + 6}{5x^3 + 6x^2 + 2} = 0, \text{ as the}$$

\swarrow degree 2
 \nwarrow degree 3

degree of numerator is strictly less than the degree of denominator.

As $\lim_{x \rightarrow +\infty} f(x) = 0$ (which is finite), the graph f admits horizontal

asymptote at $+\infty$ $\boxed{\text{the line } y = 0}$ (the x -axis).

\bullet $\boxed{\text{Similarly at } -\infty.}$

$$b) \lim_{x \rightarrow +\infty} \frac{x^3 + 3x^2}{6x^3 + x} = \frac{1}{6}, \text{ as the degrees are equal and the}$$

\swarrow degree 3
 \nwarrow degree 3

dominant coefficients are 1 for the numerator and respectively

6 for the denominator.

As $\lim_{x \rightarrow +\infty} f(x) = \frac{1}{6}$ (which is finite), the graph f admits horizontal

(11)

asymptote at $+\infty$ the line $y = \frac{1}{6}$.

• Similarly at $-\infty$.

c) $\lim_{x \rightarrow +\infty} \frac{21 + 4e^{4x}}{7 + 2e^{4x}}$

change of variable
 $e^{4x} = y$

(as $x \rightarrow \infty$, we have
also $y \rightarrow \infty$)

$\lim_{y \rightarrow \infty} \frac{21 + 4y}{7 + 2y} = \frac{4}{2} = 2$
degree 1 (pointing to $21 + 4y$)
degree 1 (pointing to $7 + 2y$)

As $\lim_{x \rightarrow +\infty} f(x) = 2$ (finite number) \Rightarrow the line $y = 2$ represents

horizontal asymptote for graph f at $+\infty$.

• $\lim_{x \rightarrow -\infty} \frac{21 + 4e^{4x}}{7 + 2e^{4x}}$

change of variable
 $e^{4x} = y$
 $\lim_{y \rightarrow 0} \frac{21 + 4y}{7 + 2y} = \frac{21 + 4 \cdot 0}{7 + 2 \cdot 0} = 3$

(as $x \rightarrow -\infty$, we have
 $e^{4x} \rightarrow 0$)

As $\lim_{x \rightarrow -\infty} f(x) = 3$ (finite number) \Rightarrow the line $y = 3$ represents

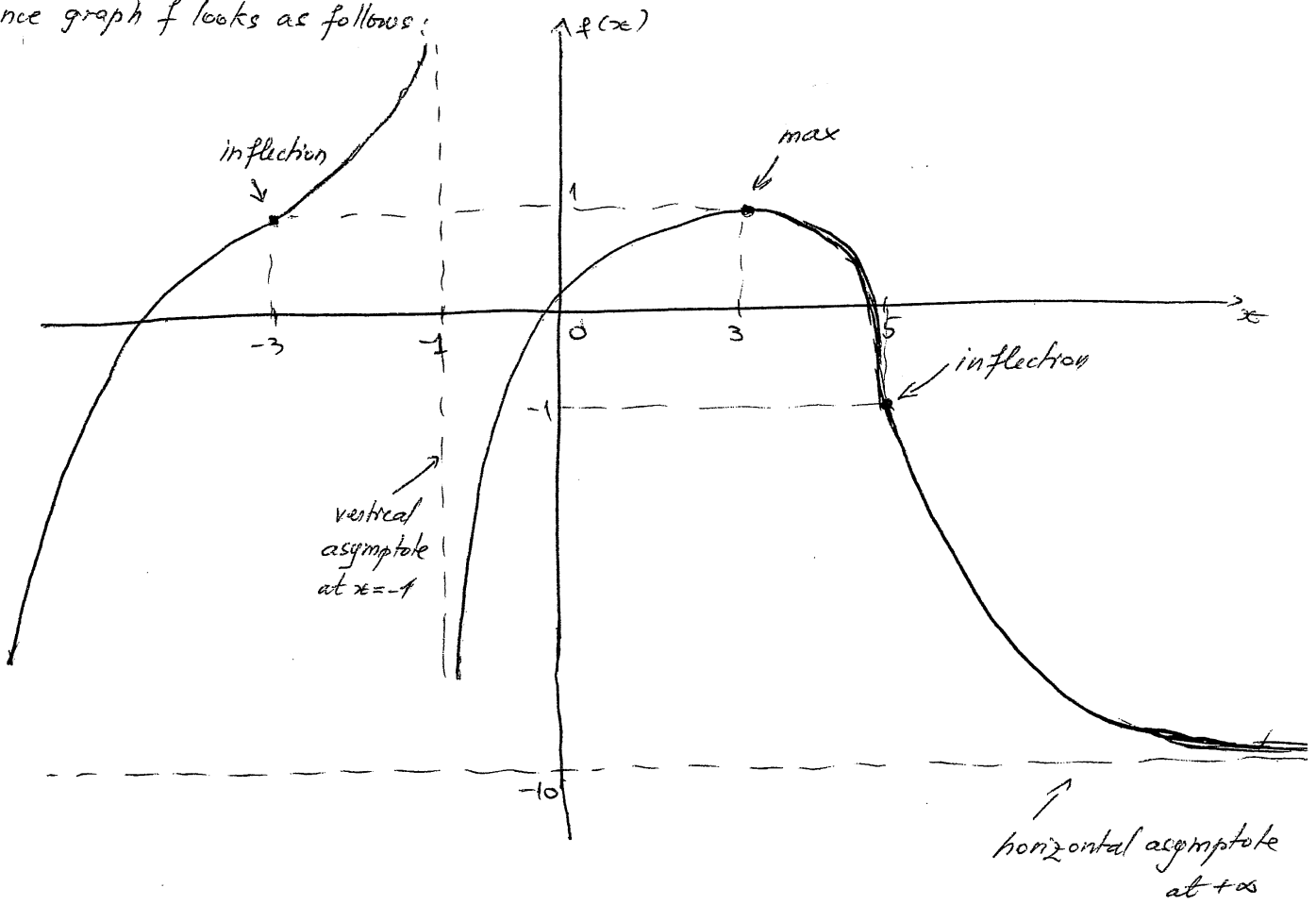
horizontal asymptote for graph f at $-\infty$.

(12) Let's concentrate the given info in the following table :

(12)

x	$-\infty$	-3	-1	3	5	$+\infty$
$f'(x)$		$+$	$+$	$+$	0	$-$
$f''(x)$		$-$	0	$+$	$-$	$-$
$f(x)$	$-\infty$	\nearrow inflection	\nearrow undefined	\nearrow max	\searrow inflection	\searrow -10

Hence graph f looks as follows:

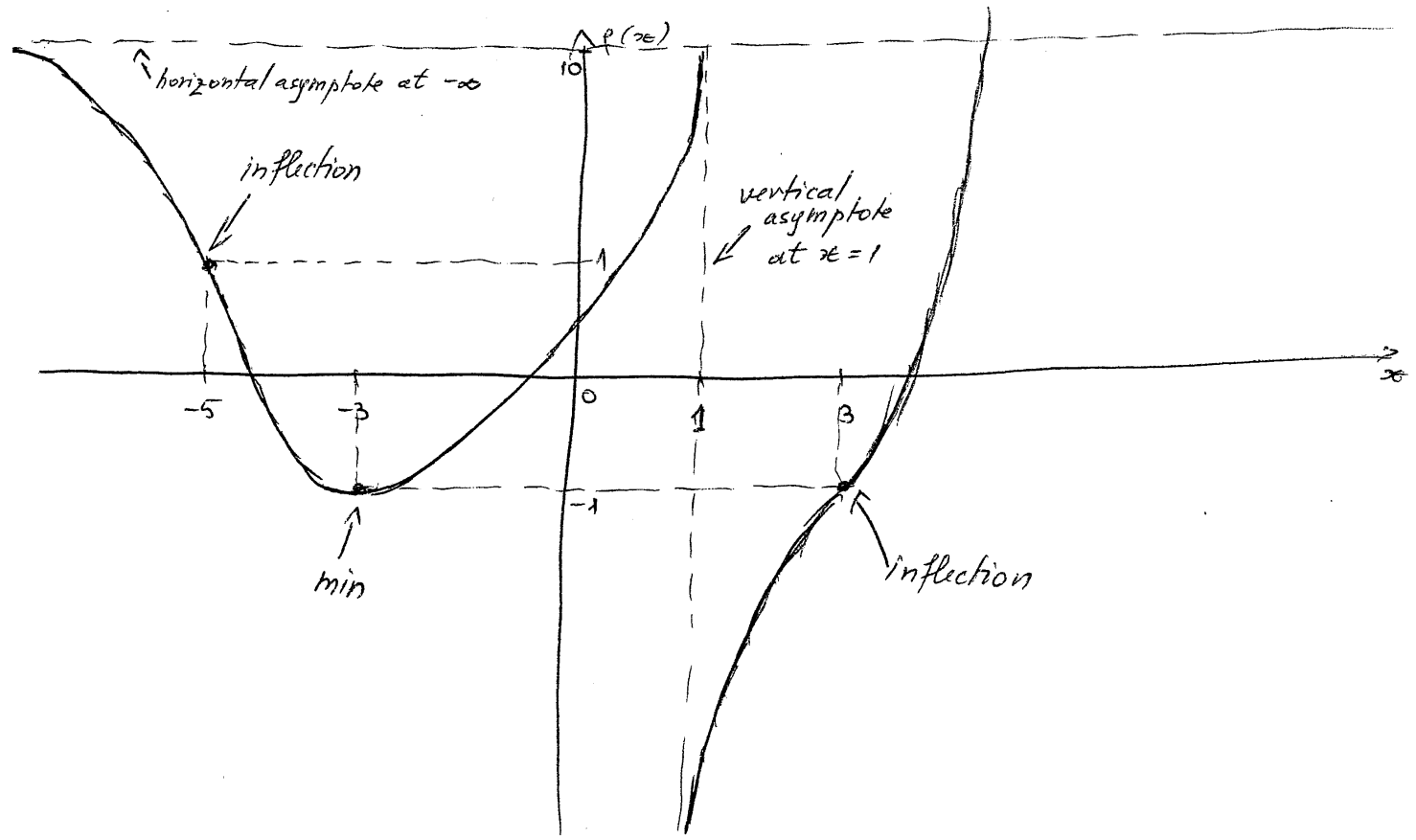


(13) Let's concentrate the given info in the following table:

13

x	$-\infty$	-5	-3	1	3	$+\infty$		
$f'(x)$		-	-	0	+	+	+	
$f''(x)$		-	0	+	+	-	0	+
$f(x)$	10	\searrow inflexion	\searrow 1 inflexion	\searrow -1 min	\nearrow undefined	\nearrow -1 inflexion	\nearrow $+\infty$	

Hence the graph of f looks as follows:



14

a) Domain $f = \mathbb{R}$ (all reals)

b) x -intercept: $f(x) = 0 \iff x^2(x^2 + 8x + 18) = 0 \iff \boxed{x=0}$.

no real roots ($\Delta < 0$);
is > 0 for all real x

y -intercept: $f(0) = 10$

14

c) As Domain $f = \mathbb{R}$, we'll have no vertical asymptotes
 As $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = +\infty$ (not finite), we won't
have horizontal asymptotes.

d) $f'(x) = 4x^3 + 24x^2 + 36x = 4x(x^2 + 6x + 9) = \boxed{4x(x+3)^2}$

$f'(x) = 0$ for $\boxed{x = -3 \text{ and } x = 0}$

x	$-\infty$	-3	0	$+\infty$
$f'(x)$		$-$	0	$+$
$f(x)$		\searrow $f(-3)$	\searrow $f(0)$ min	\nearrow

f is increasing where $f' > 0$, hence on $(0, \infty)$.

f is decreasing where $f' < 0$, hence on $(-\infty, -3) \cup (-3, 0)$

Local extrema are where $f'(x) = 0$ and changes sign, so at $\boxed{x = 0}$, which is a relative minimum.

e) $f''(x) = 4 [x \cdot (x+3)^2]' = 4 [x' \cdot (x+3)^2 + x \cdot ((x+3)^2)'] =$
↑
product rule

$= 4 [1 \cdot (x+3)^2 + x \cdot 2(x+3)^1 \cdot (x+3)'] = 4 [(x+3)^2 + 2x(x+3)] =$
↑
due to chain rule

$= 4(x+3) [(x+3) + 2x] = 4(x+3)(3x+3) = \boxed{12(x+3)(x+1)}$

$f''(x) = 0$ for $\boxed{x = -3 \text{ and } x = -1}$

15

x	$-\infty$		-3		-1		$+\infty$
$f''(x)$		+	0	-	0	+	
$f(x)$		⌒	$f(-3)$ inflection	⌒	$f(-1)$ inflection	⌒	

f is concave up where $f'' > 0$, so on $(-\infty, -3) \cup (-1, +\infty)$.

f is concave down where $f'' < 0$, so on $(-3, -1)$.

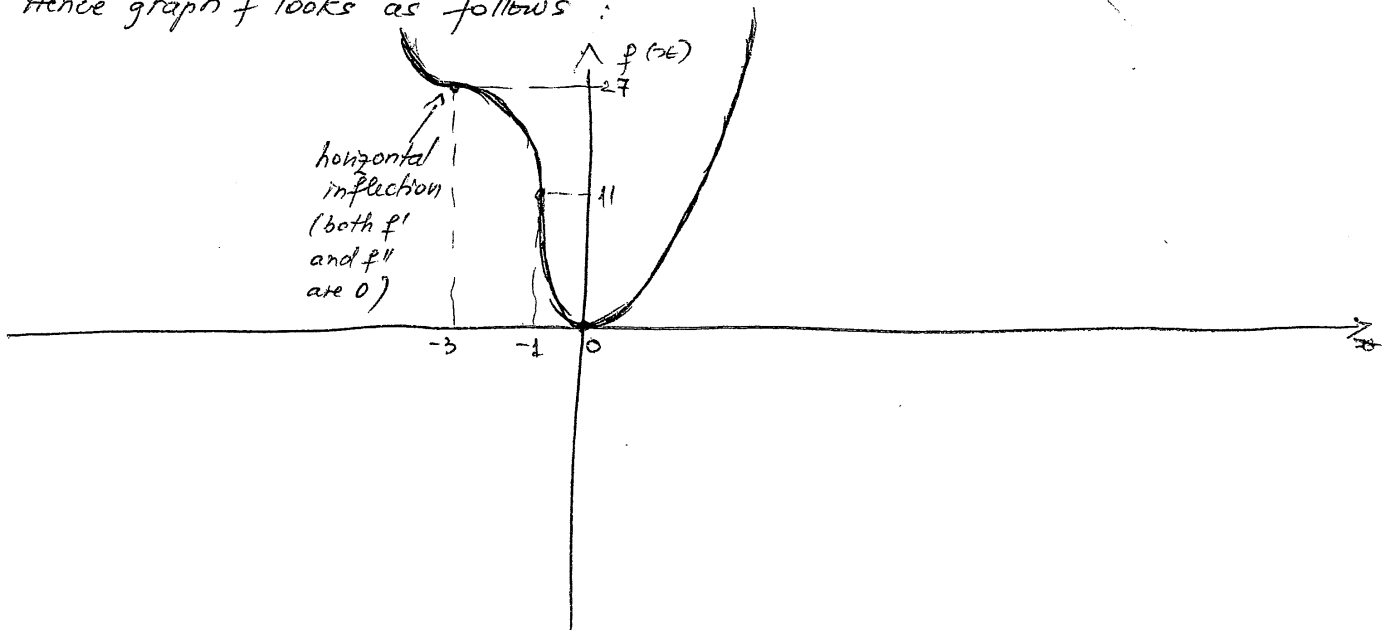
Inflection points are where $f''(x) = 0$ and changes sign, so at

$x = -3$ and $x = -1$.

f) Let's concentrate the above two tables in one single table:

x	$-\infty$		-3		-1		0		$+\infty$
$f'(x)$		-	0	-		-	0	+	
$f''(x)$		+	0	-	0	+		+	
$f(x)$	$+\infty$	⌒	$f(-3)=27$ inflection	⌒	$f(-1)=11$ inflection	⌒	$f(0)=0$ min	⌒	$+\infty$

Hence graph f looks as follows:



16

15

a) Existence condition: $x^2 - 4 \neq 0$.

Equivalently: $x \neq \pm 2$.

So $\boxed{\text{Domain } f = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)}$.

b) x -intercept: $f(x) = 0 \Leftrightarrow \frac{x}{x^2 - 4} = 0 \Leftrightarrow \boxed{x = 0}$.

y -intercept: $f(0) = \boxed{0}$.

c) Horizontal asymptotes:

$\lim_{x \rightarrow \pm \infty} f(x) = \lim_{x \rightarrow \pm \infty} \frac{\overset{\text{degree 1}}{x}}{\underset{\text{degree 2}}{x^2 - 4}} = 0 \Rightarrow$ The $\boxed{\text{line } y = 0}$

is $\boxed{\text{horizontal asymptote at } \pm \infty}$.

Vertical asymptotes:

$$\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} \frac{x}{x^2 - 4} = \frac{-2}{0^+} = -\infty$$

$$\lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} \frac{x}{x^2 - 4} = \frac{-2}{0^-} = +\infty$$

\Rightarrow $\boxed{\text{the line } x = -2}$
is vertical
asymptote
for graph f

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \frac{x}{x^2 - 4} = \frac{2}{0^-} = -\infty$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} \frac{x}{x^2 - 4} = \frac{2}{0^+} = +\infty$$

\Rightarrow $\boxed{\text{the line } x = 2}$ is
vertical asymptote
for graph f

d) $f'(x) = \frac{x'(x^2-4) - x(x^2-4)'}{(x^2-4)^2} = \frac{1 \cdot (x^2-4) - x \cdot (2x)}{(x^2-4)^2} =$

↑
quotient rule

$= -\frac{x^2+4}{(x^2-4)^2}$ ← always < 0 on its domain

f' never becomes 0 on its domain.

e) $f''(x) = (-1) \cdot \frac{(x^2+4)'(x^2-4)^2 - (x^2+4)[(x^2-4)^2]'}{(x^2-4)^4} =$

↑
quotient rule

↓
due to chain rule

$= (-1) \cdot \frac{2x(x^2-4)^2 - (x^2+4) \cdot 2(x^2-4) \cdot 2x}{(x^2-4)^4}$ after simplifications

$= 2 \frac{x(x^2+12)}{(x^2-4)^3}$ ← Becomes 0 only for $x=0$.

f) Let's concentrate in the following table the behavior of f' , f'' and f :

x	$-\infty$	-2	0	2	$+\infty$
$f'(x)$	$-$	$-$	$-$	$-$	$-$
$f''(x)$	$+$	$+$	0	$-$	$+$
$f(x)$	0	$+\infty$	0	$+\infty$	0
	\curvearrowright	\curvearrowright	inflection	\curvearrowright	\curvearrowright

Hence graph f looks like this :

18

