Stretches

1. *Weighing Cubes.* You have 6 cubes that look the same, but one is slightly heavier than the others. How many weighings on a balance are required to determine the heavy cube?

   Show each case starting with breaking it up into two sets of 3, two sets of 2, two sets of 1.

   Solution 1. *Take two sets of two. Weigh them against each other. What are the two possibilities? Suppose they weigh the same. Since only two cubes remain to be weighed, it will take one more weighing to determine the heavy cube. If the two sets of two do not weigh the same, take the heavier set. Since only two cubes are being weighed, it will take one more weighing to determine the heavy cube.*

2. You have 8 cubes that look the same, but one is slightly heavier than the others. How many weighings on a balance are required to determine the heavy cube?

   Solution 2. *Take two sets of three. Weigh them against each other. What are the two possibilities? Suppose they weigh the same. Since only two cubes remain to be weighed, it will take one more weighing to determine the heavy cube. If the two sets of the two do not weigh the same, take the heavier set. Since only two cubes are being weighed, it will take one more weighing to determine the heavy cube.*

Simplify it Today and next Tuesday we will learn about the *Simplify it* strategy. This is probably the most common strategy I use when I do my math homework and, as you will see, it will be a very powerful tool for problem solving. We discussed this strategy briefly during the first week of school, so let me remind you of it's usefulness.

Simplify it
If a problem is too difficult, we may do a simpler version of the problem to come up with a solution.
We may break the problem into smaller problems.
We sometimes need to and can make simplifying assumptions.

We will go over several problems as a class and in groups to see how Simplifying a problem may help us. Let us consider a problem we have already done to remind us how we've used simplification to solve a problem. So, to see it in action, let us review the Heap of Beans game and remind you how we used the *Simplify it* strategy to solve the problem. Recall the Heap of Beans rules.

You have a pile of 16 beans. Each person takes turns taking bean(s). A turn consists of taking 1,2, or 3 beans from the pile. The winner is the one who takes the last bean(s).

So what was a way to simplify this problem? We saw that if we were to consider the game with only four beans, the problem became trivial.
So it's clear that it is sometimes the case we can simplify a problem so that the solution becomes obvious. Now let's consider another problem where we will need to simplify the problem.

Problem 1. *On a strip with 11 squares in a line, place a quarter heads up on each of the first 5 squares, and a quarter tails up on each of the last 5 squares. The allowable moves are to move a coin one step forward (toward the opposite end) into an empty space and to hop over an opposing coin into an empty space. No backward moves or double jumps are allowed. The goal is for all tokens to migrate to the other side and change places.*

Solution 3. *We will try to brute force the game. Let's just try out random moves and see if the problem is too difficult to solve. Okay, since the problem seems too difficult, let's try the strategy of simplifying the problem. What are some ways we might want to simplify this problem? Let's try the game with just three squares. We see the problem is trivial. Okay, let's try the game with 5 squares. Again, the solution is clear after a few attempts. Okay, so do we think we have a solution for the original problem of 11 squares?*

The next problem we will work on is called the *Handshakes* problem. We will work on this problem as a class. I will try to guide you to the solution by giving you hints. So let's look at the problem at hand.

Problem 2. ***Handshakes.*** *In a train station waiting room you find yourself waiting for the train along with 15 other travelers. Everyone in the room decides that it is a good idea to become acquainted by shaking hands with everyone else in the room. How many handshakes will take place?*

Solution 4. *So, this problem might seem difficult at first, but we will see that we can in fact simplify it. Let's start off with a smaller group of people. Let's say that instead of being with 15 other travelers, there is 1 other traveler. How many handshakes will happen? Now let's say there are 2 other travelers. How many handshakes? Okay, so let's construct a table.*

Num of other People	***Num of Handshakes***
1	*1*
2	*3=1+2*
3	*6=1+2+3*
4	*10=1+2+3+4*
5	*15=1+2+3+4+5*
6	*21=1+2+3+4+5+6*
k	$\frac{k(k+1)}{2}=1+2+3+\cdots+k$

That's right, ladies and gentlemen, the number of handshakes if there were k *other people in the room is the sum of the first* k *whole numbers. Let's try to explain this. Say there is one other person. Then we have exactly one hand shake. Now if one more person walks into the room, there has to be two more handshakes and no more. That is, there is now three handshake. Now if yet one more person walks into the room, there has to be three more handshakes and no more, as the others have already shaken hands. Therefore, there are six handshakes.*

We will now consider another problem that, if not simplified, becomes practically impossible. The problem is *Telescoping Sums.*

Problem 3. ***Telescoping Sums*** *What is the sum of the following series*

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{100\times 101}?$$

Solution 5. *So we clearly need to do some simplification for this problem. I'm sure no one would want to do this summation by hand, so let's consider how we might simplify this problem. What are some ways we might consider simplifying the problem? Okay, so what we are going to do is use a table to find the sum for a similar sum of a fewer terms. One term just gives us* $1/2$*. If we did two terms, that would be* $1/2+1/6=2/3$*. Okay, so let's fill out a table and see if we can find a pattern.*

Term(s)	***Sum***
1	*1/2*
2	*1/2+1/6=2/3*
3	*1/2+1/6+1/12=3/4*
4	*1/2+1/6+1/12+1/20=4/5*
5	*1/2+1/6+1/12+1/20+1/30=5/6*

Okay, so we see a clear pattern here. After summing up the n *terms, the sum seems to add up to* $n/(n+1)$*. That is, if we sum up* 5 *terms, we get* $5/6$*. If we sum up* 10 *terms, we get* $10/11$*. Therefore, you should be comfortable with the answer that summing up* 100 *terms gives us* $100/101$*.*
Okay, so here is a bonus question. What happens if we continued this on forever? What if we continued adding the $1/(n\times(n+1))$ *terms to this sum? Will the sum be finite or add up to infinity?*
The answer is that the sum add up to 1*. Let's see why this is the case. We find that*

$$\frac{1}{n}-\frac{1}{n+1}=\frac{n+1}{n(n+1)}-\frac{1}{n(n+1)}=\frac{1}{n(n+1)}.$$

From this, if we were to continually add terms indefinitely, we find that our sum looks like

$$(\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) + \cdots.$$

We see that every term, except the first one, the 1*, cancels with another. So the* $-\frac{1}{2}$ *cancels with the* $\frac{1}{2}$*, the* $-\frac{1}{3}$ *cancels with the* $\frac{1}{3}$*, and so on. With this, and a little more rigor, we would find the sum adds up to* 1*.*

The next problem we will consider will be a bit different than the other problems we have considered. Here, when we simplify the problem, we will make appropriate assumptions. This is a very good technique for real world problems when you need to come up with a tentative or approximate solution to a problem at hand. The problem we will consider is called *The Commuter*.

Problem 4. *A commuter rides the train to and from work each day. Her husband meets her at the train station and drives her home. One day the commuter leaves work early, catches a different train and arrives at the station one hour ahead of schedule. It being a nice day, she decides to walk toward home. Somewhere along the way she meets her husband, driving from home to pick her up at the usual time. She gets in the car and they drive back, arriving 20 minutes earlier than normal. How long was the commuter walking?*

Solution 6.

The next problem we will consider is your next homework assignment. Again, this problem will require assumptions to be made to simplify the problem. We will go over the assumptions needed so you can do the problem.

Problem 5. *You pull into the mall on a Saturday shopping trip only to find all the spaces full. Instead of driving around looking for a parking spot, you poise your car at the end of a row, commanding 16 parking spaces. As soon as one of those 16 spaces opens up, you will get to park. How long do you expect to wait?*